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Simple theoretical and experimental study of convection with some geophysical applications and analogies

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A plane horizontal layer of a fluid with depth d is considered into which heat is introduced. Within the Boussinesq approximation an exact expression is obtained for the efficiency of convection γ in transforming the rate of heat supplied into the generation of kinetic energy. It agrees with results of numerical and laboratory experiments whose data can be used to calculate this value. In laboratory experiments γ is usually of order 10^{-8} to 10^{-6}. Using this quantity estimates are obtained for the r.m.s. velocity of convective motions $\overline{u}$ and their time scale $\tau = d/\overline{u}$ for a regime where viscosity is important. These estimates agree with the results of a number of previous numerical experiments over a wide range of Rayleigh, R, and Prandtl numbers. Dimensionless convection equations normalized by these scales show the existence of thermal boundary layers and of almost isothermal regions within the bulk of the fluid. From this, the main regimes of heat transfer follow immediately: the Nusselt number $N \sim (R-R_{cr})^{\frac{1}{2}}$ for moderate R and $N \sim R^{\frac{1}{3}}$ for sufficiently large R.

A number of simple experiments have been carried out to measure $\overline{u}$ and τ for convection in water. Their results confirm the theoretical dependences of $\overline{u}$ and τ on external parameters and show that a smooth transition region exists from the viscous regime of convection to the more fully-developed turbulent one. The latter regime is considered by a scaling analysis. The results are compared with the author's measurements and other experimental data.

Similarly density convection is considered which arises by the separation of a medium into light and heavy fractions. Differences and analogies with the thermal convection are established. Elementary experiments confirm qualitatively the predicted dependences for $\overline{u}$ and τ. Applications of the results obtained are briefly discussed for studies of heat and mass transfer in the ocean and of convection in the Earth's mantle.

In the last section some general properties are considered for various forced flows, convection, turbulence and some types of atmospheric circulation, that allow one to formulate a 'rule' of the fastest response, which asserts that the kinetic energy of a fluid system is of order of the supplied power times the fastest relaxation time which the system possesses.

1. Introduction

A vast literature is devoted to the consideration of convection, but nevertheless owing to the complex non-linear nature of the phenomenon the problem is still far from its complete solution. Each new problem, slightly different in its setting, requires

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new analytic attempts, extensive numerical computations or sophisticated measurements. Theoretically and experimentally convection has been studied mainly by prescribing the temperature difference on surfaces bounding the fluid. A lesser amount of work deals with the case when the heat flux is defined at a boundary as occurs in most natural phenomena. Besides, the determination of the heat flux due to convection is an important, and often the sole, aim of laboratory studies, and as a result this dependence is now known reasonably well. The present work concentrates on this second case, allowing one to derive methodically and very simply a number of general and useful results. Many known facts of convection theory are obtained as limiting cases. This leads to the possibility of understanding better what is generic for convective phenomena in general.

The central result of this study is the derivation (see § 2) of an expression for the efficiency of convection in transforming the rate of heat supplied into the rate of generation of kinetic energy of convective motions. Within the Boussinesq approximation it is exact and of simple algebraic form for a plane horizontal layer and does not depend on the nature of the motion, i.e. whether it is laminar or turbulent, whether the layer is at rest as a whole or rotating. It depends on the Nusselt number N but asymptotically the dependence vanishes. This concept unifies all types of convection, at least in horizontal layers.

In a steady state the generation rate is equal to the dissipation rate of kinetic energy in the motion. From here the dependence of the r.m.s. velocity $\overline{u}$ on external parameters is estimated for a viscous regime, including numerical coefficients for two- and three-dimensional cases. This estimate is confirmed by data of several numerical experiments on convection over a broad range of Rayleigh and Prandtl numbers. The temperature equation normalized by the turnover time scale $\tau = d/\overline{u}$ has the inverse Peclet number multiplying ΔT which rather quickly becomes small with the growth of the Rayleigh number R, implying the formation of thermal boundary layers. From here it is easily established that for moderate supercritical Rayleigh numbers $N \sim (R - R_{\text{cr}})^{\frac{1}{4}}$ and for larger R, such that $4N \gg 1$, the heat transfer is described by $N \sim R^{\frac{1}{3}}$.

For the determination of the extent of the viscous regime and for checking the predicted regularities in $\overline{u}$ and τ for three-dimensional convection at large R, several simple experiments have been carried out. They confirm these regularities for the horizontal velocity component and τ up to $R \sim 4 \times 10^7$. The results show that the transition from a viscous regime to a turbulent one is very diffuse in the sense of the dependence of $\overline{u}$ and τ on the external parameters (§§ 3 and 4).

Next, a regime of developed turbulent convection is considered by a scaling analysis of the convection equations together with the available experimental results on velocity measurements. It is shown that for very large R the rate of growth of N with R may be slowed down and suggests a possible asymptotic regime as $N \sim R^{\frac{1}{5}}$ (§ 5).

The remaining sections (6–8) are devoted to a study of convection in a fluid layer where a separation into heavy and lighter fractions occurs, to the application of the results obtained to some problems of oceanography, to the motions of lithospheric plates and to discussion (and classification) of some general properties of various forced motions. Portions of these results have been described in short notes (Golitsyn 1977*a*, *b*, 1978) but here a full and concise discussion of the theory and experiments is presented which is more general and complete than that reported previously.

Convection of incompressible viscous fluid is described by the Boussinesq equations which are

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p+\alpha g T'+\nu\Delta\mathbf{u}, \tag{1.1}$$

$$\nabla.\mathbf{u} = 0, \tag{1.2}$$

and

$$\frac{\partial T}{\partial t}+(\mathbf{u}.\nabla)\,T = k\Delta T+\frac{1}{c_p}(q+\epsilon), \tag{1.3}$$

where $\mathbf{u}$ is velocity, p pressure, ρ density, α thermal expansion coefficient, T' deviation of temperature from its equilibrium distribution in the absence of motions, g gravitational acceleration. For simplicity the kinematic viscosity ν and thermal conductivity k are assumed constant. In the energy equation (1.3) q is the rate of heat production by thermal sources per unit mass within the fluid and the rate of dissipative heating

$$\epsilon = \nu\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)^2. \tag{1.4}$$

Non-slip boundary conditions for the velocity must be added; for the plane horizontal layer the conditions for temperature are

$$\rho c_p k(\partial T/\partial z) = -f \quad \text{at} \quad z = 0, \quad \text{say}, \tag{1.5}$$

and

$$T = T_0 \quad \text{at} \quad z = d, \quad \text{say}, \tag{1.6}$$

where d is the height of the layer, f the heat flux, c_p the specific heat at constant pressure. Sometimes, instead of (1.1)–(1 3) the equation system will be written for the vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ and the enthalpy referred to the temperature T_0

$$e = c_p(T-T_0). \tag{1.7}$$

Then instead of equations (1.1)–(1.3) we shall then have

$$d\boldsymbol{\omega}/dt-(\boldsymbol{\omega}.\nabla)\,\mathbf{u} = -H^{-1}\nabla e\times\mathbf{n}+\nu\Delta\boldsymbol{\omega}, \tag{1.8}$$

$$de/dt = k\Delta e+q+\epsilon. \tag{1.9}$$

The boundary conditions will also be somewhat simplified:

$$k\partial e/\partial z = -f/\rho \quad \text{at} \quad z = 0, \tag{1.10}$$

$$e = 0 \quad \text{at} \quad z = d.$$

In equation (1.8) $\mathbf{n} = \mathbf{g}/g$ and

$$H = c_p/\alpha g, \tag{1.11}$$

a quantity with the dimension of length which may be considered as a characteristic depth of a fluid layer stratified by gravity. For air $\alpha = 1/T$ and $H = c_p T/g = T/\gamma_a$ and for $T = 288\,\mathrm{K}$ and $\gamma_a = 9{\cdot}8\,\mathrm{K\,km^{-1}}$ we have $H = 32\,\mathrm{km}$. For water at $t = 20\,^{\circ}\mathrm{C}$, $\alpha = 2\times10^{-4}\,\mathrm{K^{-1}}$, $c_p = 4{\cdot}2\times10^3\,\mathrm{J\,kg^{-1}\,K^{-1}}$ and $H = 2000\,\mathrm{km}$. For the Earth's upper mantle after McKenzie, Roberts & Weiss (1974), $\alpha = 2\times10^{-5}\,\mathrm{K^{-1}}$, $c_p = 1{\cdot}2\times10^3\,\mathrm{J\,kg^{-1}\,K^{-1}}$, and $H = 6000\,\mathrm{km}$. As is shown in particular by Hewitt, McKenzie & Weiss (1975) the Boussinesq approximation is valid for $d \ll H$.

2. Energetics of convection

A preliminary analysis of the energy balance for convection in a horizontal layer has been carried out by Lipps (1976). Here we generalize his results for the case of an arbitrary distribution of heat sources within the layer which can be useful for geophysical applications and continue his analysis. Take the scalar product of (1.1) with **u**, average it horizontally for an infinite plane layer, and integrate in height from 0 to d. The resulting energy balance equation is

$$dK/dt = G-D, \tag{2.1}$$

where

$$K = \frac{\rho}{2}\int_0^d \langle u^2+v^2+w^2\rangle\, dz \tag{2.2}$$

is the kinetic energy of a unit column and angular brackets denote horizontal averaging, so that

$$G = \alpha\rho g\int_0^d \langle wT'\rangle\, dz \tag{2.3}$$

is the rate of generation of kinetic energy from potential energy of the fluid, and

$$D = \rho\nu\int_0^d \langle \mathbf{u}\,.\,\Delta\mathbf{u}\rangle\, dz \tag{2.4}$$

is the rate of dissipation of kinetic energy of convective motion owing to viscosity. In a steady state $dK/dt = 0$ and

$$G = D. \tag{2.5}$$

Let f_1 be the heat flux introduced into the layer at the lower boundary. If there are heat sources inside the layer with intensity $q(z)$, the total heat flux at the level z is on average equal to

$$f(z) = f_1+\rho\int_0^z q(z)\, dz. \tag{2.6}$$

On the other hand the total mean heat flux in a steady state is (see, e.g. Kraichnan 1962)

$$f(z) = \rho c_p(-k\langle dT/dz\rangle+\langle wT'\rangle). \tag{2.7}$$

Equating the right-hand sides of these two expressions and integrating in height from 0 to d, then taking into account (2.3) and (1.10) one obtains

$$f_1 d+\rho\int_0^d dz\int_0^z q(z')\, dz' = \rho c_p k\Delta T+HG, \tag{2.8}$$

where $\Delta T = T_1-T_0$ the temperature difference between the lower and upper boundaries. From here, after some manipulations, one may derive an expression for the convection efficiency in transforming the heat power supplied to the layer into the rate of generation of kinetic energy

$$\gamma = \frac{G}{f} = \frac{d}{H}\frac{N_m-1}{N_m} = \frac{\alpha g d}{c_p}(1-N_m^{-1}), \tag{2.9}$$

where the modified Nusselt number is specified by

$$N_m = \frac{df(d)}{\rho c_p k\Delta T}\left\{1-\frac{\rho}{fd}\int_0^d dz\int_z^d q(z')\, dz'\right\}. \tag{2.10}$$

The case $q(z) = q = \text{const.}$ has been considered in detail for the upper mantle in numerical experiments by McKenzie *et al.* (1974) and Hewitt *et al.* (1975). In this case, equation (2.10) simplifies to

$$N_m = \frac{df(d)}{\rho c_p k \Delta T}(1 - \tfrac{1}{2}\beta), \tag{2.11}$$

$$\beta = \rho q d/(f_1 + \rho q d). \tag{2.12}$$

The parameter β is the ratio of the heat power generated within the layer to the total heat power supplied to the layer (per unit column). When $\beta = 1$ all heat is generated within the fluid, while when $\beta = 0$ the heat is introduced only from below and instead of (2.11) or (2.10) we have the usual Nusselt number

$$N = fd/\rho c_p k \Delta T. \tag{2.13}$$

For $q = 0$ we have from (2.8) and (2.13)

$$G = \alpha g \rho k \Delta T(N - 1). \tag{2.14}$$

This result (in a non-dimensional form) has been obtained by Lipps (1976).

For large Nusselt numbers, $N \gg 1$, it follows from (2.9) that

$$\gamma \cong \gamma_0 = \alpha g d/c_p = d/H.$$

This expression for the limiting efficiency of convection was, evidently, first obtained by Lliboutry (1972) from a consideration of the energy transformations in convection. It has been also obtained by Hewitt *et al.* (1975) from an analysis of the energy and entropy balance (these authors apparently did not know the Lliboutry paper). With insignificant corrections the last derivation has been reproduced by this author (Golitsyn 1977*b*, who was at the time also unaware of Lliboutry's work). We see that this last formula is valid only asymptotically when $N_m \gg 1$.

Our formula is exact by its derivation for a horizontal plane fluid layer for any type of convection in it. The only approximation is the Boussinesq one, from which it follows that it is valid when $\gamma_0 \ll 1$. At first glance this is not of great importance because it contains the Nusselt number which has to be determined separately. But the behaviour of N with Rayleigh and Prandtl numbers is now known reasonably well; besides, in §3 we present a simplified theory which will give the dependence $N(R)$ for a broad range of R. One should also remember that the efficiency becomes practically independent of N rather quickly with increasing R. In any case the formula (2.9) or its simplified versions for $q = 0$ may be used as a simple and efficient check of numerical or laboratory experiments on convection which we shall discuss now. Unfortunately not many papers on the experiments have all the necessary information for such a check but I was able to find two papers with computational results and one with suitable laboratory data.

The first was the paper by Hewitt *et al.* (1975) where results were presented of computations of the total kinetic energy dissipation rate for a fluid with parameters appropriate to the upper mantle in a square cell with size $d = 700\,\text{km}$ for $\beta = 0$, $\frac{1}{2}$ and 1 [see equation (2.12)] and for a rather wide range of Rayleigh numbers. In a steady state the dissipation is equal to the generation, so that their results may be

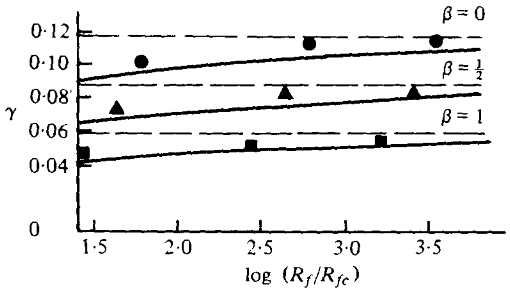

FIGURE 1. Results of computations of the convection efficiency γ by Hewitt *et al.* (1975). ●, – – –, the limiting values of $\gamma = \gamma_0 = d/H$; ——, calculation using equation (2.9).

compared with (2.9) taking into account (2.11). In figure 1 the dependence of γ is shown as a function of the logarithm of the ratio of the flux Rayleigh number

$$R_f = \alpha g f d^4 / c_p k^2 \rho \nu \tag{2.15}$$

to the critical Rayleigh number $R_{f\mathrm{cr}}$. Horizontal lines are asymptotic values of $\gamma = \gamma_0 = (d/H)(1-\frac{1}{2}\beta)$ at $d/H = 700/6000 = 0{\cdot}117$. Full circles, triangles and squares are the results of numerical computations. This part of the figure reproduces figure 3 of Hewitt *et al.* (1975). The thick curves are calculated using (2.9) and (2.11) and the dependence

$$N_m = 1{\cdot}6\,(R_f/R_{f\mathrm{cr}})^{\frac{1}{4}} \tag{2·16}$$

obtained by McKenzie *et al.* (1974) in similar computations. One should note, however, that the dependence (2.16) is presented there only for the case $\beta = 0$. Nevertheless the qualitative and quantitative agreement between (2.9) and numerical results is good. A small systematic lowering of the theoretical dots compared with the numerical ones can be probably understood as a result of a slight overestimate of the integral dissipation in the computations owing to side boundaries and/or computational viscosity or some non-stationarity of convection at very large Prandtl numbers (McKenzie *et al.* 1974). Nevertheless, it is interesting to note that the last circle at the upper right of figure 1 is practically on the line $\gamma_0 = 0{\cdot}117$.

Of all the numerical experiments, the generation of kinetic energy has been calculated directly only by Lipps (1976) for three-dimensional convection in air ($P = 0{\cdot}7$) at several values of the Rayleigh numbers up to $R = 25000$. Because he used non-dimensional equations it is not possible to extract directly from his results the values of γ. However one may check equation (2.14), which in the usual non-dimensional variables (velocity scaled by k/d, time by d^2/k) is [see Lipps (1976), equation (8)]

$$G' = PR(N-1). \tag{2.14'}$$

Lipps does not himself present the results of a numerical check of this relation. On figure 13 of his paper there are averaged profiles of specific (per unit volume) generation of kinetic energy G' with height for $R = 4000$, 9000 and 25000. Their graphical integration over vertical gives G'. Substitution of these values found for G' into (2.14'), together with corresponding values of R and N, shows that this relation is satisfied within an accuracy of a few per cent. This small discrepancy is evidently connected with the errors of graphical integration at the coarse scale of figure 13. That this is the

d (cm)	10	12	18
R	$6 \cdot 3 \times 10^5$	$2 \cdot 5 \times 10^6$	1×10^7
T (°C)	9·5	21·6	25·8
N	5·7	9	14
$\overline{G'}$	$2 \cdot 5 \times 10^6$	16×10^6	88×10^6
γ	$2 \cdot 8 \times 10^{-6}$	$3 \cdot 6 \times 10^{-6}$	$5 \cdot 6 \times 10^{-6}$
γ_{exp}	$2 \cdot 6 \times 10^{-6}$	$3 \cdot 8 \times 10^{-6}$	$5 \cdot 2 \times 10^{-6}$

TABLE 1. Experimental results by Deardorff & Willis (1967) and comparison between theoretical and experimental values of γ.

case may be shown by using his table 2 where he presents computed values of all quantities entering (2.14′) for $R = 6500$ for two runs, F and $F1$ in his notation. For the case F he gives $G'_{\text{comp}} = 4910$ at $N = 2 \cdot 08$ and (2.14′) gives 4914. For the case $F1$, $G_{\text{comp}} = 4780$ at $N = 2 \cdot 05$ and (2.14′) gives 4778. The difference is evidently due to rounding off errors.

Direct determinations of the rate of kinetic energy generation has been performed from laboratory measurements only by Deardorff & Willis (1967) who present a very large number of measured properties of convective motions. They studied convection in the air at a mean temperature of 20 °C and at pressures slightly less than atmospheric. Their results allow the possibility of determining the efficiencies of convection γ and of comparison with those calculated from equation (2.9).

In table 1 some parameters of their apparatus are presented together with the experimental results needed here, the values of $\overline{G'}$ calculated from (2.9) and values γ_{exp} calculated from measured values of $\overline{G'}$. The measured values of $\overline{G'}$ are presented in non-dimensional form. They normalized it by k^3/d^4 because the velocities were normalized by k/d and temperature by ΔT. The height-averaged generation $\overline{G'}$ can be calculated using profiles of specific generation (per unit volume) presented in their figures 17–19. The total dimensional value of the generation in unit air column is equal to

$$G = \rho \overline{G'} k^3 d^{-3};$$

the heat flux f can be calculated from (2·13) since the Nusselt number is known together with other necessary parameters. As a result

$$\gamma_{\text{exp}} = \overline{G'} k^2 / N d^2 c_p \, \Delta T.$$

The value of the thermal diffusivity in these experiments was equal to $0 \cdot 263\ \text{cm}^2\,\text{s}^{-1}$. As one sees in table 1 the value of γ_{exp} and the theoretical γ agree with an accuracy of better than 10%. One should note that the accuracy of measurements of the heat flux and some other parameters in these experiments estimated by the authors was itself about 10%, and the agreement between the two sets of γ should be considered excellent.

It is interesting to study the behaviour of the efficiency γ on the Rayleigh number

$$R = \alpha g \Delta T d^3 / k\nu \tag{2.17}$$

near its critical value. For this one should know the dependence $N(R)$ in this region.

Note that the usual Rayleigh number R is connected with the Rayleigh flux number according to the definitions (2.13), (2.15) and (2.17):

$$R_f = NR. \tag{2.18}$$

A very detailed study of the dependence $N(R)$ near R_{cr} has been carried out by Schlüter, Lortz & Busse (1965) the results of which were later checked numerically by Clever & Busse (1974). For roll convection with upper and lower rigid boundaries of the layer Schlüter *et al.* found the following relationship:

$$R(N-1)/(R-R_{cr}) = (0{\cdot}69942-0{\cdot}00472P^{-1}+0{\cdot}00832P^{-2})^{-1}, \tag{2.19}$$

and for hexagonal cells

$$R(N-1)/(R-R_{cr}) = (0{\cdot}89360+0{\cdot}04959P^{-1}+0{\cdot}06787P^{-2})^{-1}. \tag{2.20}$$

Here $P = \nu/k$ is the Prandtl number. When P is not very small the dependence on it is weak and both formulas can be represented approximately as

$$N-1 = b(1-R_{cr}/R) = b(1-r^{-1}), \tag{2.21}$$

where $r = R/R_{cr}$, $b = 1{\cdot}43$ for straight horizontal rolls and $b = 1{\cdot}12$ for hexagons. The structure of the relationship (2.21) was confirmed for $r \lesssim 3$ by rather precise measurements by Koschmieder & Pallas (1974) who studied heat transfer by convective concentric rolls in a cylindrical cavity, in which they found $b = 1{\cdot}48$. The dependence of the type (2.21) can be also obtained from two-dimensional calculations by Veronis (1966) for $r \lesssim 2$, though the value of b is found to be close to 2 which possibly can be explained by the free surface boundary conditions.

Using (2.21), equation (2.9) can be written as

$$\gamma = (d/H)(r-1)[(1+b^{-1})r-1]^{-1}$$

so that for a supercritical but not large Rayleigh number, the efficiency of the convection is a simple rational function of the Rayleigh number. Finally, if $R-R_{cr} \ll R_{cr}$ it follows from (2.21) that

$$\gamma = (d/H)b(r-1), \tag{2.22}$$

and the efficiency is a linear function of $\Delta R = R-R_{cr}$.

From numerous measurements (Jakob 1949; Kutateladze 1970) and numerical experiments we know that for a plane horizontal layer for large R

$$N = \beta_1 R^{\frac{1}{3}} \tag{2.23}$$

where $\beta_1 = 0{\cdot}1$–$0{\cdot}2$ depending on the type of the boundary conditions. We see that the convection efficiency is a monotonic function of the Rayleigh number approaching, rather slowly with R, a constant determined by the gravitational acceleration, the depth of the layer and the physical parameters of the fluid α and c_p.

Since an essential part of the dependence of the efficiency on the external parameters is the multiplier $1-N^{-1}$, we present its behaviour with R according to various computations and measurements on figure 2. The triangles represent computations by Lipps & Somerville (1971) at $P = 200$, circles are calculated by Lipps' (1976) data ($P = 0{\cdot}7$) and vertical series of two small dots and a line are due to Clever & Busse (1974) at $P = 7$ for three different aspect ratios. The full curve is drawn by hand as a

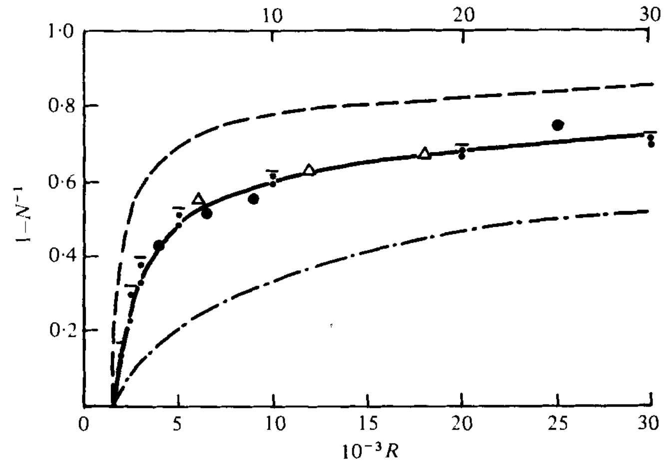

FIGURE 2. Dependence of $1-N^{-1}$ on R according to data of various authors (see text). :, CB, $P = 7$; Δ, LS, $P = 200$; ●, L, $p = 0{\cdot}7$. ---, $(1-N^{-1})^{-\frac{1}{2}}$; -·-, $P = 0{\cdot}085$.

mean of all these data and the dashed curve $(1-N^{-1})^{\frac{1}{2}}$ correspond to it. The latter is important for estimates of velocity and time scales (see § 3).

For small Prandtl numbers one may expect that the curves $1-N^{-1}$ will be lower. For instance, for the turbulent convection regime at $P \ll 1$, Kraichnan (1962) predicts that $N(R,P) \approx 0{\cdot}17(PR)^{\frac{1}{3}}$ while for $P \gtrsim 1$ he obtains $N = 0{\cdot}09R^{\frac{1}{3}}$, independent of P. The experiments by Rossby (1969) with mercury ($P = 0{\cdot}025$) agree with the idea of a substantial decrease of N with P for the same R. Using his results the lower dash-dotted curve was calculated.

The structure of the expression for the efficiency is general for domains of rather arbitrary forms. The general expression for the total kinetic energy generation rate in the volume V is (Hewitt *et al.* 1975; Golitsyn 1977*b*)

$$G = -\int_V \alpha T u_i \frac{\partial p}{\partial x_i} dV.$$

For convection where only the vertical scale of the domain is not large compared with the horizontal one and the form of the domain is not too intricate,

$$u_i \frac{\partial p}{\partial x_i} \approx w \frac{\partial p}{\partial z} = -\rho g w. \tag{2.24}$$

Implicit in this statement is the approximation that the vertical pressure gradients are large compared with horizontal ones which means that the fluid is not too far from hydrostatic equilibrium. Then taking into account (1·10) and the constancy of α, g and c_p – the usual convection approximation – we get

$$G \approx \frac{1}{H}\int_V \rho c_p w T' dV. \tag{2.25}$$

The total heat flux into the domain is

$$F = \int_S f_i dS_i, \tag{2.26}$$

where $f_i = -\rho c_p k \partial T/\partial x_i$, dS_i is the oriented element of the surface S bounding the volume in question. Then define the convective efficiency in the volume using (2.24)–(2.26) as

$$\gamma = \frac{G}{f} = \frac{1}{H}\int_V \rho c_p wT' dV \bigg/ \iint_S f_i \, dS_i.$$

In the case of a plane layer, with the aid of (2.8) and (2.10), we can obtain the earlier expression, (2.9). In the general case it is evident that

$$\gamma = (d/H) f(N) = (d/H) f_1(R, P), \tag{2.27}$$

where f and f_1 are limited functions of their arguments and d is a characteristic vertical scale of the domain (with, of course, the proviso that γ should be much less than unity for the Boussinesq approximation).

To illustrate this statement we present an estimate of the convection efficiency in a rotating fluid annulus with the external wall warmer than the internal one (Golitsyn 1977*b*),

$$\gamma \approx (d/H)\, Pe/2\pi^2 N,$$

where Pe is the Peclet number, a function in this case of the Nusselt, Taylor and thermal Rossby numbers. This approximate expression gives results which agree rather well with data of detailed computations by Williams (1971) of the regime when baroclinic waves are observed in an annulus.

A number of exact self-similar solutions are known for convection in infinite space. These include the Polhausen problem of convection near a vertical heated wall, problems by Zel'dovich on laminar and turbulent convective plumes and some others (see Landau & Lifshits 1954, § 56 with problems; Monin & Yaglom 1965, § 5.9). The characteristic vertical scale of the domain is absent in these problems and therefore expressions for the total generation of kinetic energy are formally divergent at infinity. In any event, the Boussinesq equations are applicable only for a fluid of finite vertical extent. In such cases one should use the full hydrodynamical equations or, at least, the deep convection approximation where in the continuity equation the term $wd \ln \rho_0/dz$ is preserved, where $\rho_0(z)$ is the mean density of the medium. This gives a scale height $H_0 = dz/d \ln \rho_0$. A realistic consideration is usually complicated by the non-stationarity of deep convection phenomena.

Let us consider now the dissipation of kinetic energy in convection. In a steady state, from (2.5) the average generation of kinetic energy is equal to the average dissipation. Therefore, the mean total dissipation

$$\mathrm{E} = \gamma F,$$

where F is the total heat flux introduced into fluid. For simplicity we further consider convection in a plane horizontal layer, where for a unit mass we have on average

$$\epsilon = \mathrm{E}/\rho d = \gamma f/\rho d, \tag{2.28}$$

and taking into account (2.9) at $q = 0$ we obtain

$$\epsilon = \frac{f}{\rho H}\frac{N-1}{N}. \tag{2.29}$$

For sufficiently large Nusselt numbers it follows from this that the specific dissipation does not depend on the layer depth or on the Rayleigh or Prandtl numbers but is determined only by the heat flux, gravity, and the properties of the fluid:

$$\epsilon \approx f/\rho H = \alpha g f/\rho c_p = \alpha g f', \tag{2.30}$$

where f' is the kinematic heat flux. In any case this estimate can be always used as an upper limit. Modifications of equations (2.29) and (2.30) are evident in the presence of the volume heat sources.

For a perfect gas $\alpha = 1/T$ and then (2.30) becomes

$$\epsilon = (g/T)f'. \tag{2.30$'$}$$

This formula has been obtained by Oboukhov (1960) from similarity and dimensional arguments while considering the convective regime in the atmospheric surface layer. We see that it is valid for large Rayleigh numbers such that $N \gg 1$.

3. Estimates of r.m.s. velocities and time scale of convection; regimes of heat transfer

A knowledge of the convection efficiency in the transformation of the rate of heat supplied to the rate of generation or dissipation of kinetic energy in a steady state allows one to estimate the mean (r.m.s.) velocities of convective motions. This can be done when a characteristic scale of the motion is known and when the viscosity still plays a role at this scale. An exact specification of these conditions I have not been able to define but the experiments described in § 5 show the limits of this 'viscous' regime.

Let us suppose that in the expression (1.4) for the specific dissipation all the derivatives are of the same order. In the plane case this is so if all the stream function isolines have roughly the shape of more or less concentric circles approximately equally spaced within each other (see e.g. McKenzie *et al.* 1974). Approximate $\partial v_i/\partial x_k$ by $2U/d$ where U is the velocity scale sought and $\frac{1}{2}d$ the radius of a 'circle'. In the two-dimensional case or for roll convection there are eight terms in the sum (1.4), so that $\epsilon \approx 32\nu U^2/d^2$. Taking into account (2.29) we obtain therefore that

$$U \approx \left(\frac{\epsilon}{32\nu}\right)^{\frac{1}{2}} d = \left(\frac{f}{32\mu H}\frac{N-1}{N}\right)^{\frac{1}{2}} d, \tag{3.1}$$

where $\mu = \rho\nu$ the dynamic viscosity. In the essentially three-dimensional case there are eighteen terms in (1.4) and with the same approximations for the derivatives we obtain in the denominator of (3.1), 72 instead of 32. Equation (3.1) becomes slightly more convenient if it is rewritten not for the total mean velocity U but for vertical or horizontal components of the velocity assuming that they are of the same order of magnitude. Then $U = (\overline{u}^2+\overline{w}^2)^{\frac{1}{2}} \approx 2^{\frac{1}{2}}\overline{u} \approx 2^{\frac{1}{2}}\overline{w}$ and one obtains from (3.1),

$$\overline{u} \approx \overline{w} \approx \frac{1}{a}\left(\frac{\epsilon}{\nu}\right)^{\frac{1}{2}} d = \frac{1}{a}\left(\frac{\alpha g f}{\mu c_p}\frac{N-1}{N}\right)^{\frac{1}{2}} d, \tag{3.2}$$

where $a \approx 8$ or 12 depending on the two- or three-dimensionality of the convection. Of course this estimate is quite rough but one can expect the value of a to be appreciably more than unity, and more in the three-dimensional case than in the two-dimensional one. In § 4 the value of a is determined from results of several numerical and special laboratory experiments; the results show that our crude estimate is indeed not far from reality.

The dependence $U^2 \approx (2\epsilon/a\nu)\, d^2$ is quite similar in its structure to the formula for the mean-square relative velocity of fluid particles in a fully developed turbulent flow separated by a distance d less than Kolmogorov's microscale

$$\eta = (\nu^3/\epsilon)^{\frac{1}{4}}. \tag{3.3}$$

In the Kolmogorov (1941) theory there are exact formulas for the longitudinal and lateral structure functions (mean squares of velocity component differences taken at two points separated by the distance d):

$$D_{ll}(d) = 2D_{nn}(d) = (2\epsilon/15\nu)\, d^2. \tag{3.4}$$

The difference between expressions (3.1) and (3.4) occurs only in the numerical factors, and, indeed, such a structure for expressions of mean velocities is common to all forced flows when viscosity is essential. Another example of such a flow will be given in § 6 and a discussion of some general properties of forced flows is given in § 8.

When a convective regime is slightly supercritical so that $(R/R_{\mathrm{cr}}) - 1 = r - 1 \ll 1$, then from (2.22) we have

$$1 - N^{-1} = b(r-1)$$

and using (3.2) we estimate the velocity scale as

$$\overline{u} \approx \frac{1}{a}\left[\frac{f}{\mu H} b(r-1)\right]^{\frac{1}{2}} d, \tag{3.5}$$

which implies that $\overline{u} \sim (R - R_{\mathrm{cr}})^{\frac{1}{2}}$. This fact has been used many times in theoretical studies of convection at moderately supercritical Rayleigh numbers. Equation (2.21) holds when $r \lesssim 2$ and $\overline{u}$ is a fractionally-rational function of R. For sufficiently large R when $N \gg 1$, equation (3.2) simplifies and

$$\overline{u} \approx \overline{w} \approx \frac{1}{a}\left(\frac{fd^2}{\mu H}\right)^{\frac{1}{2}} = \left(\frac{\gamma_0 f d}{\mu}\right)^{\frac{1}{2}} = \frac{1}{a}\left(\frac{\alpha g f}{\mu c_p}\right)^{\frac{1}{2}} d. \tag{3.6}$$

With the velocity scale, one may find the time scale of convective motions $\tau = d/\overline{u}$, a turn-over time. From (3.2) one obtains

$$\tau = a\left(\frac{\mu c_p}{\alpha g f}\frac{N}{N-1}\right)^{\frac{1}{2}}. \tag{3.7}$$

For slightly supercritical convective regime $\tau \propto (r-1)^{\frac{1}{2}}$ and when $N \gg 1$,

$$\tau = a(\mu c_p/\alpha g f)^{\frac{1}{2}} = a(\mu/fH)^{\frac{1}{2}}. \tag{3.8}$$

This displays the rather fundamental fact that the convection time scale does not depend on the depth for developed convection when N is large. Further, since $[N/(N-1)]^{\frac{1}{2}}$ depends rather weakly on R and therefore on d (see figure 2), this statement is approximately true for all $R/R_{\mathrm{cr}} > 5$, say. This fact has not yet received proper attention though for large R it apparently has been contained in papers by Howard (1966), McKenzie *et al.* (1974) and a formula of the structure (3.8) has been obtained by Foster (1971) by treating the results of numerical experiments.

Estimates of the velocities (3.2) and times (3.7) in terms of Rayleigh numbers R_f and R are also useful. Using the definitions (2.15) and (2.17), together with (2.22), we obtain

$$\bar{u} \approx \frac{1}{a}\left(R_f \frac{N-1}{N}\right)^{\frac{1}{2}} \frac{k}{d} = \frac{1}{a}[R(N-1)]^{\frac{1}{2}} \frac{k}{d} \tag{3.9}$$

and

$$\tau \approx \frac{ad^2}{k[R_f(1-N^{-1})]^{\frac{1}{2}}} = \frac{ad^2}{k[R(N-1)]^{\frac{1}{2}}}. \tag{3.10}$$

Since the value of $(1-N^{-1})^{\frac{1}{2}}$ becomes comparable with unity rather quickly as R increases (if the Prandtl number is not too small) – see the dashed line on figure 2 – then the velocity scale (3.6) should be a good estimate of the velocity scale over a rather broad range of values of the external parameters.

The energy equation (1.3) in a state that is stationary in mean allows one to determine a temperature scale, δT, if the equation is represented as an approximate balance between advection of temperature and heat flux divergence

$$v_i \frac{\partial T}{\partial x_i} \approx -\frac{1}{\rho c_p} \frac{\partial f_i}{\partial x_i}. \tag{3.11}$$

Approximating $\partial T/\partial x_i$ as $\delta T/d$ and $\partial f/\partial x_i$ as f/d one obtains for this scale,

$$\delta T \sim \frac{f'}{\bar{u}} = \frac{f\tau}{\rho c_p d} = \frac{a}{d}\left(\frac{f\nu}{\alpha g \rho c_p}\right)^{\frac{1}{2}}. \tag{3.12}$$

Since the thermal-conductivity does not enter here, it appears that this scale corresponds to the temperature change outside of thermal boundary layers.

Let us non-dimensionalize equations (1.8) and (1.9) choosing the length scale as d but not yet making a specific choice for the velocity and temperature scales U_0 and T_0. Then we obtain

$$Re\,[d\boldsymbol{\omega}/dt - (\boldsymbol{\omega}.\nabla)\mathbf{v}] = -C\nabla T \times \mathbf{n} + \Delta\boldsymbol{\omega}, \quad \mathbf{n} = \mathbf{g}/g, \tag{3.13}$$

$$dT/dt = Pe^{-1}\Delta T + Mq + \gamma_0 \epsilon \tag{3.14}$$

with boundary conditions on temperature at $z = 0$: $\partial T/\partial z = -MPe$ (or $T = T_1/T_0$) and $T = 0$ at $z = 1$. Here and in (3.13)–(3.14) non-dimensional variables are denoted as they were previously (and later) by the dimensional ones. In this system there are 5 similarity parameters: the Reynolds number

$$Re = U_0 d/\nu, \tag{3.15}$$

$$C = \alpha T_0 g d^2/\nu U_0; \tag{3.16}$$

the Péclet number,

$$Pe = U_0 d/k = Re\,P, \tag{3.17}$$

and a measure of the thermal inertia (see Golitsyn 1970, 1973, 1977*a*)

$$M = f/\rho c_p T_0 U_0 = f\tau_0/I, \tag{3.18}$$

where $\tau_0 = d/U_0$, $I = \rho d c_p T$ the enthalpy of a unit fluid column. In the absence of heat sources and neglecting for a moment the dissipation, the temperature equation (3.14) is uniform relative to the choice of the scale T_0. If $Pe \gg 1$ a thermal boundary layer arises with thickness

$$\delta \sim Pe^{-\frac{1}{2}} d, \tag{3.19}$$

but in the bulk of the fluid the temperature changes are small. From the continuity of the heat flux in the boundary layer and in the bulk it follows that $f \approx k\Delta T/2\delta$, where $\frac{1}{2}\Delta T$ is a change of temperature at one boundary (the case of a free upper boundary needs some elaboration and will be partially considered later). From here, (3.19) and (2.15) we obtain

$$N \sim \tfrac{1}{2}Pe^{\frac{1}{2}}. \tag{3.20}$$

This almost trivial result (of little use in the absence of velocity estimates) we shall exploit extensively. Up to now we have not yet specified the choice of U_0. For a study of a viscous convection regime it is natural to take U_0 as the velocity scale $\bar{u}$ in the form (3.9). Then from (3.17) and (3.20) a relation follows between Nusselt and Rayleigh numbers:

$$N \sim \tfrac{1}{2}a^{-\frac{1}{2}}[R(N-1)]^{\frac{1}{4}}. \tag{3.21}$$

For moderately supercritical Rayleigh numbers when (2.21) holds we get from here

$$N \sim \tfrac{1}{2}a^{-\frac{1}{2}}(R-R_{\mathrm{cr}})^{\frac{1}{4}}. \tag{3.22}$$

Let us check whether the small parameter, the inverse Péclet number is really small. For $R = 2R_{\mathrm{cr}} = 2\times 1708$ (rigid boundaries), $b \approx 1{\cdot}5$ in (2.21) and $a \approx 9$ (see § 4) we get $Pe^{-1} = a[R(N-1)]^{-\frac{1}{2}} = 0{\cdot}18$. Therefore at twice the supercritical Rayleigh number one can expect an appreciable thermal boundary layer. This is confirmed by numerical experiments (see, e.g. Veronis 1966; Clever & Busse 1974). The dependence (3.22) is just an approximation of the relationship (2.21) showing that N is, also approximately, a simple rational function of R.

A limiting case is when N is large, more precisely, $(N-1)^{\frac{1}{4}} \approx N^{\frac{1}{4}}$ for which one needs $\frac{1}{4}N \ll 1$. Then from (3.21) we obtain immediately that

$$N \sim 2^{-\frac{4}{3}}a^{-\frac{2}{3}}R^{\frac{1}{3}}. \tag{3.23}$$

Both heat transfer relationships, (3.22) and (3.23), are well known from experiments (Jakob 1949; Kutateladze 1970). It is true that the first one is usually presented in the form $N \sim R^{\frac{1}{4}}$ but, because of the smallness of the exponent, the value of $(R-R_{\mathrm{cr}})^{\frac{1}{4}}$ is close to $R^{\frac{1}{4}}$. If, to the author's knowledge, the first regime has not yet been explained theoretically, the second regime has been obtained by many workers (see, e.g. Kraichnan 1962; Herring 1966; Thompson 1967). We present it solely to show that it also is contained in this simple approach which is free from some assumptions made heretofore. Another reason is that the numerical coefficient in (3.23) is surprisingly close to its experimental value. The empirical relationship of the type (3.23) is usually written as $N = \beta_1 R^{\frac{1}{3}}$, where $\beta_1 = 0{\cdot}1$–$0{\cdot}2$ depending on the type of the boundary conditions. For instance at both rigid walls of a plane fluid layer $\beta_1 = 0{\cdot}13$. We have the factor $2^{\frac{4}{3}}\,a^{-\frac{2}{3}}$ which is about $0{\cdot}1$ when $a \approx 9$ and about $0{\cdot}08$ when $a \approx 12$. Anyway the derivation shows that the coefficient can be an order of magnitude less than unity.

We see that the relationship (3.21) describes qualitatively and, to some extent, quantitatively the dependence $N = N(R)$ for $R \gtrsim 2R_{\mathrm{cr}}$ at Prandtl numbers that are not very small (otherwise equation (2.21) is not valid). For small P, i.e. for large thermal diffusivity, the Péclet number can remain small with increase of R, the thermal boundary layer is not pronounced and our considerations are hardly justified.

Of course, the Nusselt number depends also upon the horizontal scales of the convective cells but this dependence is rather weak and reveals itself mainly in a slight change of the intensity of the $N(R)$ curves without changing their shapes (see, e.g. Clever & Busse 1974, and their points on figure 2). To check this statement the relationship (3.21) was converted into an equality in such a way that it would give asymptotically $N = 0{\cdot}13R^{\frac{1}{3}}$, and the dependence $N(R)$ was calculated from it. The curve obtained was compared with similar curves of figure 10 by Willis, Deardorff & Somerville (1972) and it was found to lie approximately in the middle between the experimental points for variable non-dimensional perturbation wavelength λ and the computed curve at $\lambda = 2$ for the Rayleigh number range 6000–20000. One may therefore consider equations (3.21)–(3.23) as simple and more or less accurate estimates of the dependence of heat transfer on the Rayleigh number.

The asymptotic heat transfer regime $N \propto R^{\frac{1}{3}}$ begins rather quickly, at least in numerical experiments, such as the computations by McKenzie *et al.* (1974) where it already held for $R/R_{\text{cr}} \gtrsim 6$. It begins early in other experiments (see Herring 1963, 1964; Veronis 1966; Lipps & Somerville 1971). We have already noted that because $(1-N^{-1})^{\frac{1}{2}}$ quickly approaches unity the velocity scale (3.9) should be a good estimate for convection and then one may obtain (3.23) immediately from (3.20).

It follows from (3.17) that at the large Prandtl numbers characteristic of the Earth's mantle ($P \sim 10^{23}$), the Reynolds number $Re = Pe/P$ is small and nonlinear terms in (3.13) are insignificant. We have not yet specified the temperature scale T_0. Let us choose it in the form (3.12). Then the similarity criterion $C \equiv 1$ in (3.16) and (3.13) will not depend on any criteria. With T_0 as in (3.12) the similarity criterion $M \equiv 1$ again applies and $\gamma_0 = d/H$. Accordingly, in the absence of internal heat sources and for $Pe \gg 1$ the system (3.13)–(3.14) has no similarity parameters except Pe in (3.14), and outside thermal boundary layers for $Pe \gg 1$ the system becomes self-similar. For $4N \gg 1$ it follows from (3.17) and (3.9) that

$$Pe = R_f^{\frac{1}{2}}. \tag{3.24}$$

This self-similarity of convection for large P has an important consequence for laboratory modelling of mantle convection if one is not interested in the detailed structure of thermal boundary layers. The model flow must fulfil only the requirements $Re = R_f^{\frac{1}{2}} P^{-1} \ll 1$, $\gamma \ll 1$, $R_f^{\frac{1}{2}} \gg 1$ which are not too severe. An exact correspondence between the model and reality should be only for the boundary condition $\partial T/\partial z = -MPe = -R_f^{\frac{1}{2}}$, with our scaling. One may check that the laboratory experiments by Booker (1976) are close to fulfilling these requirements.

At the end of this section we consider briefly the connexion between our results and those already known, and also some other useful consequences. At large Nusslet numbers ($4N \gg 1$) when the regime (3.23) is valid we have from (3.9) and (3.10):

$$\bar{u} \approx \frac{R_f^{\frac{1}{2}}}{a}\frac{k}{d} \quad \text{or} \quad \bar{u} \approx \frac{\beta_1^{\frac{1}{2}}}{a} R^{\frac{2}{3}} \frac{k}{d}, \tag{3.25}$$

$$\tau \approx \frac{ad^2}{R_f^{\frac{1}{2}} k} \quad \text{or} \quad \tau \approx \frac{a}{\beta_1^{\frac{1}{2}}} R^{-\frac{2}{3}} \frac{d^2}{k}. \tag{3.26}$$

The second of the formulas (3.25) has been apparently first obtained by Turcotte & Oxburgh (1967) and the first one by McKenzie *et al.* (1974). The latter authors have also derived the first formula (3.26) and the second one is in the paper by Howard (1966). All these authors have derived their results by considering the balance of energy and vorticity but without the numerical coefficients a or β_1 which both differ from unity by an order of magnitude.

Similarly one may express the temperature scale (3.12) as

$$\delta T = aR_f^{\frac{1}{2}} \frac{k\nu}{\alpha g d^3} = a \frac{R_f^{\frac{1}{2}}}{R} \Delta T = a \left(\frac{N}{R}\right)^{\frac{1}{2}} \Delta T. \tag{3.27}$$

For $N = \beta_1 R^{\frac{1}{3}}$ we obtain from here that the temperature difference in the bulk of fluid outside the thermal boundary layers referred to the total temperature difference is

$$\frac{\delta T}{\Delta T} = a\beta_1^{\frac{1}{2}} R^{-\frac{1}{3}} \sim R_f^{-\frac{1}{4}}. \tag{3.28}$$

This shows again that the bulk of the fluid tends to isothermy with increase of the Rayleigh number and, in addition, shows the rate at which this takes place.

We also present several formulas which will be used later, valid in the regime $N \sim R^{\frac{1}{3}}$. The thickness of the thermal boundary layer δ is defined in such a way that

$$f = \rho c_p k \Delta T / 2\delta. \tag{3.29}$$

Comparing this expression with the definition of the Nusselt number (2.13) one gets $N = d/2\delta$, from which, accounting for (3.23) and (2.22), it follows that

$$\delta = (d/2\beta_1) R^{-\frac{1}{3}} = (d/2\beta_1^{\frac{3}{4}}) R_f^{-\frac{1}{4}} = (\rho k^2 \nu / 16 \beta_1^3 \alpha g f)^{\frac{1}{4}}. \tag{3.30}$$

If on a boundary the heat flux is prescribed but the temperature drop throughout boundary layer is not known it can be determined from (3.29) and (3.30) as

$$\Delta T = \frac{2f\delta}{\rho c_p k} = \left(\frac{R_f}{\beta_1}\right)^{\frac{3}{4}} \frac{k\nu}{\alpha g d^3} = \left(\frac{f}{\beta_1 \rho c_p}\right)^{\frac{3}{4}} \left(\frac{\nu}{\alpha g}\right)^{\frac{1}{4}} k^{-\frac{1}{2}}. \tag{3.31}$$

This expression describes, for instance, the temperature drop in a cold film of fluid cooling from the surface, when convection is due to instability of this film. The measurements by Katsaros *et al.* (1977) and Ginzburg, Zatsepin & Fedorov (1977) confirm well this expression.

Let us note that the scales of velocity (3.6), time (3.8) and temperature (3.12) can be also obtained in several other ways, each of which has, however, some limitations. For instance at $Re \ll 1$ and $Pe = Re\,P \gg 1$ these scales can be immediately derived from a scaling analysis of the equation system (1.8), (1.9), but without estimates of the numerical coefficients. In the appendix another derivation is presented using similarity and dimensional arguments which seem to be somewhat non-trivial and require some use of elementary group theory. It is evident that these scales should depend on the type of boundary conditions, the aspect ratio of the domain and on the distribution of internal heat sources, e.g. on the parameter β defined by (2.12).

4. Comparison with numerical and real experiments

Because of difficulties in measuring the velocity and temperature fields, many experimenters limit themselves to a quantitative exploration of only the dependence $N(R, P)$. Some exceptions are the works of Malkus (1954), Deardorff & Willis (1967) and Garron & Goldstein (1973) of which the author became aware only after writing the third draft of this paper. These papers will be discussed in § 5 because their material relates more to the turbulent convection regime. Unfortunately, because of the same lack of experimental evidence, authors of numerical papers pay most attention to checking the same dependence and to determining the mean temperature profile but data on the velocity field and its mean characteristics are rare. Some data on the velocities do exist in the paper by McKenzie *et al.* (1974) which, incidentally, gave an impetus to the studies reported here. They describe an extensive series of experiments on numerical modelling of convection in the Earth's upper mantle, in which they varied the heat flux value f over the range 10^{-4} to $0{\cdot}1\ \mathrm{W\,m^{-2}}$ (we recall that the mean geothermal flux is about $0{\cdot}06\ \mathrm{W\,m^{-2}}$). Equations (56), (61) and (62) by McKenzie *et al.* (1974) present the following computed dependences of the *maximum* values of horizontal velocity $\hat{u}$ at the upper boundary:

$$\left.\begin{aligned}\beta = 0,\ \log\hat{u} &= 0{\cdot}50\log f + 1{\cdot}91;\\ \beta = \tfrac{1}{2},\ \log\hat{u} &= 0{\cdot}49\log f + 1{\cdot}80;\\ \beta = 1,\ \log\hat{u} &= 0{\cdot}54\log f + 1{\cdot}85.\end{aligned}\right\} \tag{4.1}$$

We recall that, in virtue of (2.12), the upper line corresponds to the heat flux supplied only from below and the lower line, only from within. Here $\hat{u}$ is measured in $\mathrm{mm\,yr^{-1}}$, the density of the medium is $3{\cdot}7\ \mathrm{t\,m^{-3}}$, and $\nu = 2\times10^{17}\ \mathrm{m^2\,s^{-1}}$. Taking into account (2.11) and $d/H = 0{\cdot}117$ our formula (3.1) can be transformed for these same values of the parameters into

$$\log\hat{u} = 0{\cdot}5\log f + 1{\cdot}61 + 0{\cdot}5\log(1-\tfrac{1}{2}\beta). \tag{4.2}$$

Since equation (4.1) describes maximum velocities, while (4.2) represents the mean velocities, it appears that our result is valid not only in the sense of the dependence on the main external parameters, but also in determining the order of magnitude of the numerical coefficient $a \approx 8$ over a wide range of variation of the heat power introduced into the system.

A dependence of the type (4.1) may be also obtained using data of numerical experiments by Houston & DeBremaecker (1975) in their constant viscosity convection case. Their parameters were: $\nu = 5\times10^{17}\ \mathrm{m^2\,s^{-1}}$, $\rho = 3{\cdot}5\ \mathrm{t\,m^{-3}}$ and

$$f = 5{\cdot}94\times10^{-2}\ \mathrm{W\,m^{-2}},\quad \beta = 0{\cdot}79.$$

Under these conditions, their $\hat{u} = 16\ \mathrm{mm\,yr^{-1}}$. A substitution of these values into (3.2) allows one to determine the value of the last term in a formula like (4.1), leading to

$$\log\hat{u} = 0{\cdot}5\log f + 1{\cdot}52.$$

The difference in the last term is due mainly to the larger viscosity; two kinds of numerical experiments and our equation (3.2) are in satisfactory accord, at least for $Re \ll 1$.

In some papers on numerical experiments one can find occasional values of the velocities or mean kinetic energy of convection K together with Nusselt numbers. These quantities can be used to find the numerical coefficient a using the following expressions derived from (3.9):

$$a = \overline{w}^{-1}[R(N-1)]^{\frac{1}{2}},\quad \text{or}\quad a = [R(N-1)/K]^{\frac{1}{2}}. \tag{4.3}$$

If the present approach has any value, the magnitudes of a should be constant or should, at most, vary little.

	P		
R	0·2	1	7
3000	9·04	8·86	8·79
5000	9·25	8·92	8·84

TABLE 2. The results calculated from Clever & Busse's (1974) data.

Veronis (1966) considered two-dimensional convection by Galerkin's method. He presented only two values of the magnitude of the vertical velocity: $w = 31{\cdot}1$ at $P = 6{\cdot}8$ and $w = 32{\cdot}3$ at $P = 0{\cdot}005$ for $R = 20R_{\text{cr}}$. He considered the case with both boundaries free and therefore his $R_{\text{cr}} = 658$. The corresponding Nusselt numbers are 5·33 and 5·676. Substitution of these values into the first formula (4.3) gives $a = 7{\cdot}68$ for both Prandtl numbers (the difference is in the fourth decimal), though the Prandtl numbers themselves differ by three order of magnitude.

Clever & Busse (1974) also studied two-dimensional convection by Galerkin's method, but for rigid boundaries. Using the results of their computations presented graphically in their figure 4 as $K(P)$ and $N(P)$ for $0{\cdot}2 < P < 7$ for two Rayleigh numbers $R = 3000$ and 5000, one may calculate the values of a according to the second formula (4.3). These results are presented in table 2. There is the suggestion of a very weak increase, of the order of a percent or less, of the value a with decrease of the Prandtl number and with increase of the Rayleigh number. The estimates given here can hardly pretend to be valid with this accuracy, and in addition, some small inaccuracies are also due to the values of K and N taken from the graphs. It seems justified therefore to average the results of this table, presenting them as $a = 8{\cdot}95 \pm 0{\cdot}17$ (standard deviation) or, rounding, $a = 9{\cdot}0 \pm 0{\cdot}2$.

Very recently, Dr F. H. Busse kindly supplied me with the results of his computations (in press) on the dependences of the mean kinetic energy and the Nusselt number on the wavenumber α of the convection cells, under conditions with both boundaries free, $R = 20\,000$ and Prandtl number infinite. Busse's curves for $N(\alpha)$ and $K(\alpha)$, together with (4.3) yield values of a from 4·55 to 4·98, varying almost linearly with α over the range $2{\cdot}4 \leqslant \alpha \leqslant 4{\cdot}1$. This implies that the mean convection velocities increase slightly with increase of the wavelength of convection, as is consistent with the comments in the appendix of this paper.

The work of Lipps (1976) was found to be most valuable. In his paper, he presents his mean values of K in tables. He integrated the complete hydrodynamical equations for air ($P = 0{\cdot}7$) for six three-dimensional cases over the range of Rayleigh numbers from 4000 to 25 000 and also for two two-dimensional cases. When $R \leqslant 9000$ the convection with rigid boundaries organized itself in regular two-dimensional horizontal rolls, but for $R = 25\,000$, a regular space-time structure was not observed. All eight cases allow one to calculate values of a, using (4.3). The results of such calculations are presented in table 3, where the first column gives the type of experiment in Lipps' notation, while the columns to the right indicate the Rayleigh numbers, $N-1$ and the non-dimensional kinetic energy.

If one includes the case E of fully three-dimensional convection then the mean value of a is $9{\cdot}05 \pm 0{\cdot}30$ while, without the case E, $a = 8{\cdot}95 \pm 0{\cdot}12$. We see that the two different methods of computations, over a rather broad range of the Rayleigh and

type	R	$N-1$	K	a
A	4000	0·75	38·4	8·84
B	4000	0·88	45·5	8·80
D	9000	1·22	134·3	9·04
D-$2D$	9000	1·25	135·1	9·13
E	25000	1·94	482·3	10·03
E-$2D$	25000	1·89	503·4	9·69
$F1$	6500	1·05	86·7	8·87
$F2$	6500	1·08	86·9	8·99

TABLE 3. Values of a calculated from Lipps' (1976) data.

Prandtl numbers, give, with a good accuracy, an approximate constancy of the coefficient a which is about 9 for the rigid boundary case with an accuracy of a few per cent, surprisingly close to our rough estimate $a \approx 8$. The results of Veronis (1966), though there are only two numbers available for free boundary conditions, are also close to our estimate.

The fact that the coefficient a appears to be very nearly universal confirms our estimate (3.1) and reflects a very simple physical interpretation of the formula for the mean (dimensional) kinetic energy K of a unit column. From (2.2) and (3.2), it follows that

$$K \approx a^{-2}\gamma f \tau_\nu, \quad \tau_\nu = d^2\nu^{-1}. \tag{4.4}$$

One sees from here that the kinetic energy K is of the order (with a factor a^{-2}) of the mechanical power $G = \gamma f$ generated in the fluid by the heat supplied, times the viscous relaxation time τ_ν.

One may also calculate the Reynolds number for the convective motions.

$$Re = \frac{Pe}{P} = \frac{[R(N-1)]^{\frac{1}{2}}}{aP} = \frac{K_n^{\frac{1}{2}}}{P}. \tag{4.5}$$

where K_n is the non-dimensional kinetic energy, measured in k^2d^{-2} units. From the data of table 3, it varies from 8·8 (A) to 32·4 (E), and the question arises concerning the limits of applicability of the results obtained in terms of values of the Reynolds (or Rayleigh) number. The question was especially acute for the author because originally a formula of the type (3.1) had been obtained from similarity arguments when $Re \ll 1$ (upper mantle; Golitsyn 1977*a*). However, the derivation, presented here (also given there as an explanatory one) does not require explicitly the smallness of Re.

Even larger values of Re are obtained in analysing the results of detailed computations by Williams (1967, 1971) concerning convection energetics in rotating annuli, heated differentially. It happens that the energetic relations described here are also found there. Though the heating from side walls and rotation decrease the efficiency of convection by several times in comparison with the case of heating from below, nevertheless, if the specific dissipation is known, the mean velocities can be estimated as

$$U \approx (\epsilon/32\nu)^{\frac{1}{2}}\, d \tag{4.6}$$

for the two-dimensional or axisymmetric case (Williams 1967), and as

$$U \approx (\epsilon/72\nu)^{\frac{1}{2}}\, d \tag{4.7}$$

for the three-dimensional case when baroclinic waves appear in an annulus (Williams 1971). In fact, in the first case, the data of the direct computations allow one to deduce $U = 2{\cdot}6\,\mathrm{mm\,s^{-1}}$ and $\epsilon = 3\times10^{-3}\,\mathrm{cm^2\,s^{-3}}$. Using this value of ϵ and the annulus depth $d = 5$ cm one gets, according to (4.6), $U \approx 2{\cdot}4\,\mathrm{mm\,s^{-1}}$. For the wave regime the results of the computations yield $U = 1{\cdot}2\,\mathrm{mm\,s^{-1}}$ and $\epsilon = 1{\cdot}1\times10^{-3}\,\mathrm{cm^2\,s^{-3}}$. Substitution of this ϵ and $d = 3$ cm into (4·7) produces $U = 1{\cdot}1\,\mathrm{mm\,s^{-1}}$. More details may be found in Golitsyn (1977*b*).

The values of the Reynolds number for these two cases are 80 and 35, respectively. On the one hand, the values are large in the sense that boundary layers are relatively thin, but on the other hand the values are still sufficiently small that the flow patterns are regular and laminar though rather complicated; the viscous forces essentially determine the patterns and the velocity gradients are appreciable even in the bulk of the fluid.

In an attempt to understand why this theory works even at rather high Reynolds numbers, one may formally introduce the Kolmogorov internal microscale (3.3), if the value of ϵ can be evaluated. For Williams' first case one gets $\eta = 1{\cdot}8$ mm and for the second, $\eta = 1{\cdot}3$ mm. The ratio of the scale d to η is equal to 30 and 17, respectively. Apparently, if

$$H_k = d/\eta \sim Re^{\frac{3}{4}}, \tag{4.8}$$

the flow has a laminar or slightly irregular character and our theory may be extended to these conditions, though one must not expect similarity of the flow patterns for $Re \gtrsim 1$.

However, Williams has published detailed results only for these two cases. In order to see whether the agreement is not fortuitous (results in tables 2 and 3 were obtained much later) I began to consider the possibility of an experimental check of equations (3.1) or (3.6) for larger Reynolds numbers. Finally, two sets of experiments were carried out, one for qualitative information, and another for a complete check of the dependence of the mean velocities on the external parameters including a determination of the value of numerical coefficient a in the three-dimensional case.

The idea of the experiments was first conceived while watching the behaviour of grains and bubbles in soup standing on a slow fire. Their velocities were seen to be of the order of few centimeters per second, the right order of magnitude according to this theory. The hope was that simple measurements could be made from the trajectories of the particles which, in part, were almost rectilinear and horizontal though rather irregular in space and time; but that required only the sufficient statistics.

In the first series of experiments (carried out in my kitchen during two late evenings) the linear dependence of the mean velocities on depth was checked while other parameters remained fixed. In the first evening the 'technology' of the measurements was worked out. The preliminary results showed an approximate proportionality between $\bar{u}$ and d. In the second evening a series of measurements was performed and the results are presented in figure 3.

Here is a description of the 'technology' of the experiments which any reader may carry out by himself with a stop-watch and couple of hours of spare time. To measure distance I had drawn with a ball-point pen, a 1-cm grid at the bottom of white enamelled saucepan 20 cm in diameter. The pan was in a water bath, consisting of a wide frying pan, the saucepan standing on three small pieces of wooden rod 1 cm high. The depth of the water in pan was about 2 cm, and the entire construction was

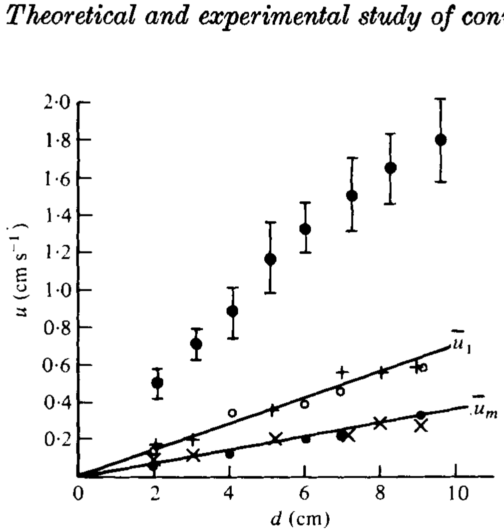

FIGURE 3. Dependence of the mean horizontal velocity on depth. Upper dots with bars (dispersion) for thermal convection, lower dots for density convection in gasified mineral water: $u_1 = d/T_1$, maximal velocities; $u_m = d/T_m$, mean velocities. Circles and crosses correspond to water from different bottles.

on a slow constant gas burner. On a nearby stove was a large teapot on a small burner with water of about the same temperature, which was used to add water to the pan to compensate for evaporation and to change the level of the water in the saucepan. The temperature in the bulk of water in the saucepan was measured by a laboratory mercury thermometer and during the entire time of measurement, it was 83 ± 1 °C. It changed little during the measurements and so if one is interested only in the check of the $u \propto d$ dependence, it is not necessary to have the thermometer but one should keep all the burners constant. The depth of the water in the kettle was measured by a ruler. The choice of tracers was a problem at first, but I found that almost any dry organic powdered material will serve, because, becoming wet, it has practically neutral buoyancy. In the experiment described I used tea, powdered by myself, and a dry red wild rose berry.

Most of these particles lay at the bottom, some were on the surface but some were transported within the water. Their path was observed by eye on the co-ordinate grid and the time of travel over rectilinear parts was measured by the stop-watch. In the main series, the measurements were carried out for 8 depths from 2 to 10 cm. For each layer there were about 35 individual measurements of path and time. During the time of measurement at each depth, a layer of water about 3 mm thick evaporated and the horizontal size of the points in figure 3 reflects this fact. The vertical bars in figure 3 show the dispersion which is in range of 15–20%. Through the first six points one may draw a direct line, but some deviation of the last two points can be seen. This is discussed below while considering the results of more complete experiments (see figure 4). Note that these two points correspond to Reynolds numbers $Re \gtrsim 2000$.

The other series of experiments aimed at a complete check of equation (3.6) was performed under laboratory conditions with the assistance of A. A. Grachov. The measurements of velocities were carried out as described above, but the 'technology' was somewhat more complicated. The side walls of another cylindrical vessel 16 cm in diameter were covered by asbestos for heat insulation, and instead of the water

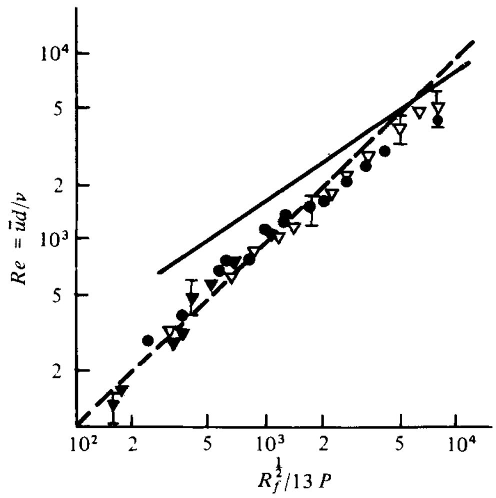

FIGURE 4. Check of the relationship (4·8) by data of measurements in water (see text). ——, $R^{\frac{1}{3}}_f$.

bath, a wet sand bath was used. The pan was standing on an electroplate with variable voltage (from 150 to 250 V). By changing the voltage and the water depth (from 2 to 7 cm) we achieved a rather broad interval of Rayleigh numbers. In the process of measurements the wetness of sand was also changed, affecting the heat transfer to the kettle.

The heat flux was determined in the following way. After the temperature in the vessel had become constant, it was weighed on an electronic scale with an accuracy of 0·1 g. The initial depth of water was determined by the difference in weight between the filled and empty vessel and by its internal diameter. Then during the next 10 minutes or so, we did about 30–35 individual measurements using the tea particles. After this, we again checked the temperature of water and made a new weighing. Knowing the time between two successive weighings, the amount of water evaporated and the heat of evaporation, one can find the heat flux f_e spent for evaporation, which was found to be about 80–90% of the total heat flux introduced into the kettle. The rest consisted of heat flux radiated from the water surface and the flux causing convection in the air above the water. The first part was estimated by the formula

$$f_r = \sigma(T_w^4 - T_a^4), \tag{4.9}$$

where σ is the Stefan–Boltzmann constant and indices w and a relate to the water and air (also the ceiling and wall of the room).

The flux causing convection in the air was estimated by the relationship $N = 0{\cdot}13\,Ra^{\frac{1}{3}}$, wherefrom

$$f_c = 0{\cdot}13\rho c_p\, k^{\frac{2}{3}}(\Delta T)^{\frac{4}{3}}\,(\alpha g/\nu)^{\frac{1}{3}}. \tag{4.10}$$

This formula describes the heat transfer from a heated plate into a medium, the mean temperature of which is $\Delta T = T_w - T_a$ lower. The formula as well as equation (4.9) requires a knowledge of the water surface temperature T_{ws} which is lower than the mean water temperature. The value of T_{ws} was found by the following method of successive approximations. It is based on equation (3.31), which expresses the drop of temperature in the thermal boundary layer in terms of the heat flux and material parameters of the fluid. In the first approximation we used the value of the heat flux spent on the evaporation, f_e. Then equation (3.31) determined the temperature drop ΔT_1 in the cold film with some underestimate. The value of $T_{ws1} = T_w - \Delta T_1$ then determined the water surface temperature T_{ws} with some overestimate. The value of T_{ws1} was substituted into equations (4.9) and (4.10) and the total heat flux

$$f_1 = f_e + f_{r1} + f_{c1}$$

was calculated. The value of f_1 was then used again in (3.31) and so on. The procedure converged very rapidly and to determine the water surface temperature with an accuracy of 1 °C one iteration was found to be enough.

Material constants of the water and their dependences on temperature (which varied in individual series from 55 to 75 °C) were taken from tables. The dependence (3.6) of the mean velocity on external parameters (the Nusselt number, of course, was large) can be presented as a relationship between Reynolds, Rayleigh and Prandtl numbers as

$$Re = R_f^{\frac{1}{3}} (aP)^{-1}. \tag{4.11}$$

The results of our experimental check of this relationship are given in figure 4. The Reynolds number was determined using the horizontal velocity component measured, the depth of the fluid and the viscosity at the mean temperature of the water. The Rayleigh flux number R_f was calculated using the measured heat flux f and other material constants at the same temperature. The value of the coefficient a was first found as a mean value of a_i calculated for each set of individual measurements from $a_i = (\alpha g f / \mu c_p)^{\frac{1}{3}} / u_i$ for $Re < 1500$. Though this is not an optimal way to determine the value of a, it produced $a = 13 \pm 2$. Only the dark points were taken into account, the circles representing measurements by A. A. Grachov and triangles those by the author. Evidently, the personality of the experimenter does not influence the results of the measurements. The dashed line correspond to $a = 13$. The regression line between the values of $R_f^{\frac{1}{3}}/13P$ and Re was calculated and the regression coefficient was found to be 1·023. Taking this value into account and also the dispersion of the coefficient, the final value of a from these measurements is equal to $12{\cdot}7 \pm 1{\cdot}4$.

Next, an attempt was made to see whether the points of my previous measurements checking the proportionality between u and d were consistent with the dependence (4.11). Since the temperature was measured and kept constant within ± 1 °C in all measurements presented in figure 3, the heat flux can be estimated using the value found for a and equation (3.6). For this estimate the first four points of figure 3 were used and the heat flux was estimated with a scatter of about 20%. All the points of figure 3 with their mean value of f were then adjusted to the relationship (4.11) and were plotted on figure 4 as light triangles. Among them there are also a few points of the first evening of my experimental activity when the temperature was also measured and was about 90 °C. The relative behaviour of the light triangles is in fair agreement with the data of later, more complete measurements.

As a whole, the data of figure 4 allow one to conclude that (3.6) or (4.11) hold up to $Re \lesssim 1500$, perhaps 2000, and the experimental estimate of the coefficient a is quite close to its crude theoretical estimate $a \approx 12$. At higher values of the Reynolds number there is a systematic deviation of the measured velocities, connected evidently with the appearance of a regime of developed turbulence, when different relationships are acting (see § 5).

There is another type of convection which has been studied extensively by numerical and laboratory experiments, the results of which also give the possibility of a check upon some of the dependencies obtained here. This is the convection within a fluid arising from cooling at the upper surface. It has been studied theoretically by Howard (1966), numerically by Foster (1971) and in laboratory by Katsaros *et al.* (1977) and Ginzburg *et al.* (1977). One of the basic results here is that the convection has a quasi-cyclic character. The major part of the cycle is occupied by the growth of a thermal boundary layer near the surface due to molecular heat exchange with the colder medium above. This layer is often called a 'cold film'. When it is heavier than the bulk of the fluid and the film becomes thick enough, cold thermals form from it, plunge down and mix the fluid rather quickly; then the process repeats itself. It is clearly important for studying the heat and salt budget in the upper ocean so it has attracted considerable attention.

The picture of the convection just described was first proposed by Howard (1966) who estimated the period of the cycle as $\tau \sim d^2k^{-1}R^{-\frac{2}{3}}$ [see equation (3.26)]. It was confirmed in numerical computations by Foster (1971) who obtained the formula (3.8) and determined the value of a to be equal to 14 according to his computations. In the work by Ginzburg *et al.* (1977) such a picture was observed experimentally. They present several durations of individual periods which agree with equation (3.8) at $a = 14$ within an accuracy of 20%. By the author's request these experiments were continued with the aim of systematic checking of equation (3.8). In the measurements the temperature was registered as a function of time at a fixed point 3 mm below the water surface for various temperature differences between water and air. The heat flux from water to air was determined calorimetrically. The range of conditions can be defined by the Rayleigh number R_f. If as a length scale one takes the depth of the tank then R_f varied between 3×10^9 and 5×10^{10}. In detail, the measurements and their results are described in a note by Ginzburg, Golitsyn & Fedorov (1979) (see also Ginzburg *et al.* 1977). Of course, the space-time pattern of the development of the thermals is rather chaotic, and so the single-point measurements of $T(t)$ allow one to estimate the convection time scale only if the measurement is long enough for sufficient statistics. The treatment of a large amount of data (the records of a total duration of about 30 hours with individual cycles being from 15 to 75 seconds) gave a rather high correlation coefficient between measured values of a and the calculated ones using (3.8), equal to 0·88. The value of the coefficient a obtained by calculating the regression line was found to be $12{\cdot}1 \pm 2{\cdot}3$ (standard deviation).

One should not exclude the possibility that such a close coincidence with the value of $a = 12{\cdot}7 \pm 1{\cdot}4$ obtained in the experiments described previously may be to some extent fortuitous, but the fact of the existence of regularities of the type (3.6), (3.8) or (4.11) over certain broad ranges of conditions, whatever the numerical coefficients are, seems to be established for various types of thermal convection. On the other hand, this coincidence bears witness to a symmetry of convective regimes with respect to simultaneous changes in signs for heat flux and buoyancy force. That means that instability processes causing convection in the lower thermal boundary layer, where to the heat is introduced, are similar to the ones in the upper thermal boundary layer from which the heat is extracted.

5. Turbulent convection

A turbulent regime of convection sets in at very large Rayleigh or r.m.s. Reynolds numbers, when the influence of viscosity in the bulk of the fluid becomes insignificant, and the regularities found in § 3 are replaced by another set. It is in this sense that we are considering the regime of developed thermal turbulence, because in the laboratory experiments described in § 4, although the pattern of particle motions was irregular, the r.m.s. velocities still depended on the viscosity.

The turbulent regime was considered in detail by Kraichnan (1962), using mixing length concepts. Some of his results which are needed here can be reproduced very simply. Experimental data on measured convective velocities in this regime have been summarized by Garron & Goldstein (1973). As we shall see, our data for $R_e \gtrsim 1500$ qualitatively agree with them. In conclusion we show the possibility of a decrease of the rate of growth of heat transfer with Rayleigh number for very large R.

We shall start from the vorticity equation (1.8). In a steady state the main balance in the turbulent regime will be between advection terms and generation of vorticity due to buoyancy. We assume that the scale of largest energy containing eddies is comparable with the layer depth d. Then from (1.8) and equation (1.9), written as

$$u_i \,\partial e/\partial x_i \sim \partial(f_i/\rho)/\partial x_i,$$

one can obtain the following scale estimates for the velocity and enthalpy gradient:

$$U = (fd/\rho H)^{\frac{1}{3}} = (\gamma_0 f/\rho)^{\frac{1}{3}} = (\alpha g d f/\rho c_p)^{\frac{1}{3}}, \tag{5.1}$$

$$\nabla e = (f/\rho)^{\frac{2}{3}}\, \gamma_0^{-\frac{1}{3}}\, d^{-1}, \quad \gamma_0 = d/H. \tag{5.2}$$

From the last formula, there follows an estimate of the temperature difference for the bulk of the fluid:

$$\delta T \sim (f/\rho)^{\frac{2}{3}}\, \gamma_0^{-\frac{1}{3}}\, c_p^{-1}. \tag{5.3}$$

Formulas (5.1) and (5.3) are quite similar to the formulas for the distributions of velocity and temperature in the atmosphere in free convection, which are obtained immediately if one substitutes into (5.1) and (5.3), the running height z for d and remembers that for a gas $\alpha = 1/T$, where T is characteristic temperature of the environment. Let us note that a formula of the type (5.1) was first obtained by Prandtl (1932) and of the type (5.3), by Oboukhov (1960), both being derived by similarity and dimensional arguments while here we use a scaling analysis of the equations.

The connexion between (5.1) and the theory of turbulence by Kolmogorov and Oboukhov becomes evident if one recalls that due to (2.30), $f/\rho H = \epsilon$, the rate of dissipation of kinetic energy of turbulence. Therefore for turbulent convection the dissipation is cubic in the velocities, which is intrinsic to the inertial subrange of the turbulence, while for the viscous regime the dissipation is quadratic in the velocities.

Equation (5.1) can be rewritten as $U = (\gamma_0 f M^{-1} d)^{\frac{1}{3}}$, where $M = \rho d$ is the mass of a unit column and $\gamma_0 f M^{-1} = \epsilon$. The role of the efficiency of convection $\gamma_0 = d/H$ is again evident in transforming the rate of supplied heat into the rate of generation (and dissipation) of the kinetic energy in turbulent convection.

Now we use the scales (5.1) and (5.2) to non-dimensionalize the system (1.8), (1.9).

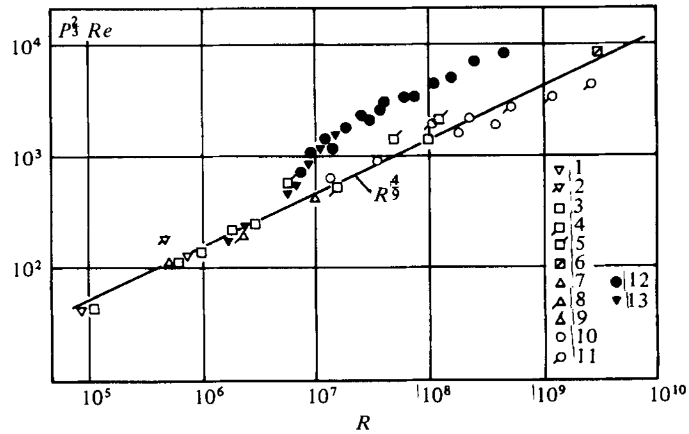

FIGURE 5. Check of the relationship $P^{\frac{2}{3}} Re \sim R^{\frac{4}{9}}$ for the turbulent regime of convection. ○, the dispersion of the vertical velocity component (after Garron & Goldstein 1973); ●, mean values of horizontal velocity component after figure 4. ▽, 1; ▽, 2; □, 3; □, 4; □, 5; ☒, 6; △, 7; △, 8; △, 9; ○, 10; ○, 11; ●, 12; ▼, 13. Points numbered 1, 2, water; 3–6, acetone, Malkus (1954); 7–9, air, Deardorff & Willis (1967); 10, 11, water, Garron & Goldstein (1973); 12, 13 water, measurements by A. A. Grachov and the author.

Neglecting for simplicity internal heat sources and heating due to viscous dissipation we obtain

$$\frac{d\boldsymbol{\omega}}{dt} - (\boldsymbol{\omega}.\nabla)\mathbf{v} = -\nabla e \times \mathbf{n} + \frac{1}{Re}\Delta\boldsymbol{\omega}, \tag{5.4}$$

$$\frac{de}{dt} = \frac{1}{Pe}\Delta T, \tag{5.5}$$

$$\frac{\partial e}{\partial z} = -Pe \quad \text{at} \quad z = 0, \quad e = 0 \quad \text{at} \quad z = 1. \tag{5.6}$$

Here all the values are non-dimensional. The relationships between Reynolds, Péclet, Rayleigh and Prandtl numbers follow from their definitions and (5.1):

$$Re = (fd/\rho H)^{\frac{1}{3}} d/\nu = a_1 P^{-\frac{2}{3}} R_f^{\frac{1}{3}}, \tag{5.7}$$

$$Pe = Re\, P = a_1 (PR_f)^{\frac{1}{3}}, \tag{5.8}$$

where a_1 is a numerical coefficient which has to be determined experimentally. Equation (5.7) corresponds to (6.12) of Kraichnan (1962), if one takes into account that in a turbulent regime $R_f = NR = \beta_1 R^{\frac{4}{3}}$.

Garron & Goldstein (1973) present a summary of measurements of vertical velocity w at mid-level for convection in a layer between rigid horizontal boundaries. The data occupy the range of Rayleigh numbers from 10^5 to 3×10^9 which, for $N = 0{\cdot}13 R^{\frac{1}{3}}$, corresponds to R_f from 6×10^6 to 6×10^{12}. The entire set of data is described more or less well by the dependence

$$P^{\frac{2}{3}} Re_v = 0{\cdot}37 R^{\frac{4}{9}}, \tag{5.9}$$

where the coefficient was determined by the present author with an accuracy of about 3%. One should note, as do Garron & Goldstein, that the dependence (5.9) is fulfilled in average for the whole interval of Rayleigh numbers under consideration, but data of individual authors lying at various parts of the interval form, as a rule, less steep subsets whose end points may deviate from (5.9) up to 60%.

Our figure 5 represents the summary by Garron & Goldstein together with our 'solid' points from figure 4. The straight line corresponds to (5.9). We see that for $R \gtrsim 3 \times 10^7$ our points are parallel to the line $0{\cdot}37R^{\frac{4}{9}}$ while for smaller R, they form a steeper sequence ($\sim R^{\frac{2}{3}}$ or $\sim R_f^{\frac{1}{2}}$). Note a small scattering of our points at $R \gtrsim 3 \times 10^7$: they deviate (in the r.m.s. sense) from the line $P^{\frac{2}{3}}Re_k = 0{\cdot}99R^{\frac{4}{9}}$ by no more than few per cent.

In a comparison of the numerical coefficients, 0·37 and 0·99, one has to consider the following circumstances. First, a three-dimensional flow, if it is isotropic, the mean horizontal component measured by us is $2^{\frac{1}{2}} = 1{\cdot}41$ times larger than the vertical one; this makes the difference between numerical coefficients less than two-fold. Second, equation (5.9) describes the results of experiments with both rigid walls while we have a free upper surface. It is known that a free surface always causes increased velocities and heat transfer (see, e.g. Herring 1963, 1964) which makes a difference between the coefficients natural. Finally, we cannot exclude the possibility that in visual measurements, an eye tends, purely unconsciously, to select particles with somewhat enhanced velocities, though we both (Grachov and I) tried by all means to avoid this. An objective instrumental check would be desirable, but nevertheless, the small scatter of points measured by the two of us and their consistent $R^{\frac{4}{9}}$ dependence does give some credence to our measurements.

There is another aspect requiring further instrumental study. At $R \lesssim 3 \times 10^7$ the values Re_h (horizontal) are definitely following the dependence (4.11) while r.m.s. fluctuations in w seem better to be described by (5.9) down to $R \sim 10^5$. There is the impression that regularities of the developed turbulent regime begin to act appreciably earlier for the vertical velocity component than for the horizontal one.

For the horizontal component one can introduce an 'index of the development of turbulence' as

$$I = \frac{R_f^{\frac{1}{2}} P^{\frac{2}{3}}}{12{\cdot}7P\,.\,1{\cdot}96R_f^{\frac{1}{3}}} = 0{\cdot}04P^{-\frac{1}{3}}R_f^{\frac{1}{6}} = 0{\cdot}03P^{-\frac{1}{3}}R^{\frac{2}{9}}. \tag{5.10}$$

If $I > 1$ the regime of convection can be considered as fully turbulent in the sense of its dependence on the external parameters; otherwise the relationships of the viscous regime are appropriate. However one sees again from equation (5.10) that the transition from one regime to another one is very smooth because of the small exponent in Rayleigh number in (5.10). The transition interval depends also on Prandtl number and for small P the turbulence starts earlier with R than for large P. This statement is in a qualitative agreement with observations by Rossby (1969) who noted that the convection in mercury ($P = 0{\cdot}025$) seemed to be turbulent much earlier with R than in water or, moreover, in an oil with $P = 200$. It would be of interest to study this question quantitatively.

It is interesting to compare our estimates of convective velocities with an estimate often used (see, e.g. Veronis 1966) which is believed to be an upper limit:

$$w^2 \leqslant \alpha g \Delta T d. \tag{5.11}$$

This estimate is obtained by the assumption that the potential energy released during the ascent of a fluid particle is more than or equal to the kinetic energy acquired by the particle. It will be shown that this is not always the case. In the viscous interval using (3.9) and (2.15) we obtain an inequality from (5.11)

$$N-1 \leqslant a^2 P. \tag{5.12}$$

Because $a^2 \sim 10^2$ the inequality is usually satisfied easily, but the case of small P causes suspicion. The computations by Veronis (1966) in fact supply a counter-example to (5.12). In our § 3 we have calculated $a = 7{\cdot}68$ at $P = 0{\cdot}005$ and $N = 5{\cdot}676$. With these values the right-hand side of (5.12) is almost 20 times smaller than the left hand side!

This fact shows that juxtaposing the released potential energy and the kinetic energy is not an evident thing, but one should consider the total energy cycle. If we call the right-hand side of (5.12) the available potential energy A (a rigorous definition of this concept has been given by Lipps 1976; one finds that the value of $\alpha g \Delta T d$ is an upper bound to A), then the equations for K and A do not provide any restriction on the relation between K and A, but they do for the rates of transition between these types of energy. The smallness of the Prandtl, number implies the relative smallness of the kinematic viscosity, and therefore the viscous relaxation time $\tau_\nu = d^2/\nu$ is relatively large. Then equation (4.4) $K \approx \gamma a^{-2} f \tau_\nu$, valid in the viscous regime, means that the kinetic energy may be large even relative to available potential energy, which is built up by thermal factors and in particular by the large heat conductivity. The last provides a quick restoration of potential energy within the fluid at the expense of heat reservoirs. Because of this small relaxation time, the potential energy can maintain a much larger kinetic energy of convection in fluids with small P.

For $P \gtrsim 1$ the inequality (5.11) is too loose. For example, consider the data due to Lipps in our table 4. For $R = 4000$ we have $N-1 = 0{\cdot}75$ and $a = 8{\cdot}8$, which means from (5·12) that $0{\cdot}75 < 8{\cdot}8^2 \times 0{\cdot}7 = 54$. For $R = 25\,000$ (case E) we similarly obtain $1{\cdot}89 < 70$. We see that the estimate (5.11) is either too loose (at $P \gtrsim 1$), or invalid. Nature is basically simple, but we should not always expect to get a right answer using a naive approach.

In the conclusion of this section, we will briefly discuss what could produce equation (3.20): $N \sim Pe^{\frac{1}{2}}$; or with our scaling, (5.8): $Pe = (R_f P)^{\frac{1}{3}}$. From here and from the definition $R_f = NR$ we obtain

$$N \sim (PR_f)^{\frac{1}{6}} \sim P^{\frac{1}{5}} R^{\frac{1}{5}}. \tag{5.13}$$

If such a regime of heat transfer exists it would mean breaking the 'principle' of locality of the heat transfer owing to convection, according to which the heat flux f through the layer [and the thickness of the thermal boundary layer δ, see (3.30)] should not depend on the total depth of the layer. Invoking this principle gives the usual explanation of the law $N \sim R^{\frac{1}{3}}$, or $f \sim \Delta T^{\frac{4}{3}}$. For the dependence (5.13) one gets $f \sim d^{-\frac{2}{5}} \Delta T^{\frac{6}{5}}$. Whether such a decrease of the heat transfer growth rate with R exists could be determined only experimentally. Exponents smaller than $\frac{1}{3}$ have been obtained by many: for instance, Garron & Goldstein (1973) obtained $N = 0{\cdot}13\, R^{0{\cdot}293}$ in water in the range $1{\cdot}3 \times 10^7 < R < 3{\cdot}3 \times 10^9$. Note that Kraichnan (1962) predicted, vice versa, an increase of the dependence $N(R)$ with R compared with $N \sim R^{\frac{1}{3}}$ for very large Rayleigh numbers. To the author's knowledge such an increase has never been observed.

In this connexion experiments measuring the heat transfer at $R > 10^{10}$, say, would be highly desirable. As far as the author knows such experiments for a plane horizontal layer have not yet been reported. In any case the dependence (5.13) should be considered only as a possible ultimate asymptotic for very large R. In the turbulent regime one might expect the existence of an intermediate zone, or boundary layer, where a temperature gradient is maintained by convection so that $N \sim R^{\frac{1}{3}}$, but in the bulk of the fluid equations (5.1)–(5.3) hold.

For reference purposes we note that the dependencies $N \sim R^{\frac{1}{3}}$ and $\delta \sim d/N \sim dR^{-\frac{1}{3}}$ have been also obtained by McKenzie *et al.* (1974) from different arguments for the convection in a layer where all heat is supplied from within and these dependencies have been confirmed by their computations. However the computed velocities followed the regularities of the viscous regime, see the last equation (4.1). This is an example of the existence of a hybrid regime.

6. Density convection

The physical reason for thermal convection is the expansion of fluid particles which, when heated, become lighter than their environment or heavier when cooled. But fluid particles may become lighter or heavier in other ways. For instance, in haline convection (Foster 1968), the instability arises from the formation of a heavier surface film owing to the increased salinity of the water by ice formation. Some geophysicists believe (Artyushkov 1968; Sorokhtin 1974; Keondjan & Monin 1977) that convection in the Earth's mantle is caused by differentiation of the mantle material, in which a heavier fraction is descending and a lighter one is arising. It is believed that this process is taking place at the mantle–liquid core interface, but one can only speculate on whether the differentiation is at a molecular or some macroscopic level. In such an uncertain situation only the simplest phenomenological approach is justified.

The analogy known between processes of heat and mass exchange allows one to use many results of thermal convection for the density convection case. However certain specifics of density convection require an accurate translation of this analogy and redefinition of some concepts. This will be done in this section. In addition, some general consequences will be discussed and an experimental check of the theory will be presented.

We shall use the Boussinesq approximation again in this study. In the momentum equation (1.1), the only change is that ρ'/ρ_0 replaces $\alpha T'$ where ρ' is the departure of the density from its equilibrium value $\rho_0(z)$. Because the motions are slow, $\nabla.\mathbf{v} = 0$. For the density deviation, one can write the equation

$$d\rho'/dt = \rho_t + k_{\mathscr{D}}\Delta\rho', \tag{6.1}$$

where ρ_t is the rate of the density differentiation in the volume, and $k_{\mathscr{D}}$ is the coefficient of density diffusion, which can be interpreted in a manner similar to the filtration coefficient of a liquid in porous media (see, e.g. Barenblatt, Entov & Ryzhik 1972).

In addition, we have the energy equation, together with the equation of state $\rho = \rho(T)$. If the differentiation is taking place at some surface (at the lower boundary of the layer, say) then it can be described by a mass flux r_t, which is, at the same time, the rate of density change at the boundary. This mass flux should be related to the gradient of density deviation

$$r_t = k_{\mathscr{D}}\,\partial\rho'/\partial z \quad \text{at} \quad z = 0, \text{ say.} \tag{6.2}$$

The other boundary conditions can be the same as for thermal convection. The total density flux $\mathscr{M}$ at a level z consists of two parts:

$$\mathscr{M}(z) = r_t + \int_0^z \rho_t(z)\,dx. \tag{6.3}$$

Let us turn now to an analysis of the energetics of density convection. The energy balance equation (2.1) preserves its form but the mean rate of kinetic energy generation in a unit column is

$$G = \int_0^d \langle \rho' w g \rangle\, dz. \tag{6.4}$$

Now, define the non-dimensional mass flux carried by the convection, a type of Nusselt number (2.13), as

$$N_g = \mathscr{M} d / k_{\mathscr{D}} \Delta\rho, \tag{6.5}$$

where $\Delta\rho$ is the difference between the densities at the upper and lower boundaries. This definition is valid when $\rho_t = 0$ and in a steady state, $\mathscr{M} = r_t$. If $\rho_t \neq 0$ then one can define a modified Nusselt number N_{gm} in a way similar to (2.10).

The mechanical power introduced into the flow in density convection, causing the motion, is equal to $\mathscr{M}gd$ in a unit column. In fact, $g\rho'$ is the force per unit volume, $g\rho' w = g\mathscr{M}$ is the power developed by the force and $\mathscr{M}gd$ is the total mechanical power in a unit column of height d. In other words, the value of $\mathscr{M}gd$ is the rate of release of potential energy in the convection.

The efficiency of the convection, in the sense of transforming the supplied mechanical power into the rate of generation of kinetic energy, we define as

$$\gamma_g = G/\mathscr{M}gd. \tag{6.6}$$

Manipulations similar to those in § 2 give for the case $\rho_t = 0$,

$$\gamma_g = (N_g - 1)/N_g = 1 - N_g^{-1}. \tag{6.7}$$

If the differentiation occurs within the layer also, we obtain formulas of the type (2.9) and (2.10). For the case of $\rho_t =$ const. one such formula is

$$\gamma_g = (1 - \beta_g/2)(1 - N_{gm}^{-1}), \tag{6.8}$$

where

$$\beta = \rho_t d/(r_t + \rho_t d) = \rho_t d/\mathscr{M}(d) \tag{6.9}$$

is a quantity, similar to (2.12), determining the fraction of the density flux which forms within the layer, compared with the total flux which could be measured at the upper surface. The difference between equations (6.7) or (6.9) and the similar equation (2.9) is that in the latter the rate of generation was related to the rate of heat supplied, but here, it is related directly to the rate of release of potential energy.

A determination of the efficiency of density convection allows one to carry over all the results of §§ 2 and 3 concerning the thermal case. We write only the formula for the mean rate of dissipation per unit mass

$$\epsilon = \frac{G}{\rho d} = \frac{\gamma_g \mathscr{M} g}{\rho} = \frac{\mathscr{M} g}{\rho} \frac{N_g - 1}{N_g}. \tag{6.10}$$

In order to make the analogy between the two types of convection completely clear, we will now obtain an expression for the mass flux in thermal convection:

$$\mathscr{M}_t = \langle \rho' w \rangle. \tag{6.11}$$

For simplicity we consider a plane horizontal layer without volume heat sources. After integration of (6.11) over height, taking into account that $\mathscr{M}_t = \text{const.}$ and equations (2.7) and (2,13), we obtain after some rearrangements

$$\mathscr{M}_t = \frac{\alpha f}{c_p} \frac{N-1}{N}. \tag{6.12}$$

Incidentally, equation (2.9) follows immediately from here if one remembers that $\mathscr{M}_t gd$ is the rate of release of potential energy in a unit column.

If both sources of convection are present, thermal and density differentiation, one can introduce the total density flux

$$\mathscr{M}_0 = \mathscr{M} + \mathscr{M}_t. \tag{6.13}$$

The total efficiency of convection in this combined case can be determined as

$$\gamma_0 = G_0 / \mathscr{M}_0 gd.$$

Recalling the definitions (2.3) and (6.4) and taking into account (2.13) and (6.5) we obtain

$$\gamma_0 = \frac{k_{\mathscr{D}} \Delta\rho (N_g - 1) + k\alpha \Delta T (N-1)}{d\,[\mathscr{M} + \alpha f c_p^{-1} (1 - N^{-1})]}. \tag{6.14}$$

In the limit of large Nusselt numbers this expression tends to unity because in that limit, all the potential energy released is spent on the generation of kinetic energy. The main similarity criterion in estimating the relative role of the two sources of convection will be the ratio $\alpha f / c_p \mathscr{M}$.

The kinetic energy of convection is dissipated into heat. In density convection this will produce a heat flux even in the absence of direct heat sources, and this heat flux can be measured at the upper surface. If the observed heat flux is partially formed by direct heat sources, there is an upper bound to the density flux due to differentiation:

$$\mathscr{M} < f_0 / gd, \tag{6.15}$$

where f_0 is the total observed heat flux. This inequality proves to be useful in a consideration of convection in the Earth's mantle, where there are radioactive heat sources.

The specification of the analogy between thermal and density convection allows one to model one type of convection by the other. The role of the Prandtl number $P = \nu/k$ will be played by the Schmidt number $Sc = \nu/k_D$ and instead of the Rayleigh flux number we will have

$$R_m = \frac{\mathscr{M} g d^4}{\rho \nu k_{\mathscr{D}}^2} = N_g Ra_m, \tag{6.16}$$

where Ra_m is an analogue of the usual Rayleigh number constructed from the density difference. Just as in the case of equation (3.23) one may obtain for sufficiently large Nusselt number that

$$N_g \sim R_m^{\frac{1}{4}} \sim Ra_m^{\frac{1}{3}}. \tag{6.17}$$

Some very simple experiments are now described in which some consequences of the analogy were checked. An everyday example of density convection is provided by the motion within a bubbling fluid such as gasified mineral water. However, the visualization of this motion and the devising of any reproducible quantitative measurements proved to be difficult. After many trials (and wasting many bottles of mineral water) I was finally able to invent a quick check of the independence of the time scale on the fluid height [see equation (3.8)] which is valid for $N_g \gg 1$.

As the working fluid I used the mineral water 'Moskovskaya' (from a drill hole within Moscow City). It was poured into a transparent glass flask in the shape of a parallelepiped, with sides $95 \times 79 \times 37$ mm. It had been noted that adding small particles of powdered black pepper† increased bubble formation strongly and also the intensity of the motion. A significant release of gas took place under these conditions for several hours. An experiment usually occupied 20–30 minutes so the conditions could be considered as stationary. The experiment was performed as follows. First a mixture of black pepper in water, about 0·5 cm^2, was put into the flask. The mineral water was then poured into the flask to a given depth and was allowed to settle for a few minutes to allow the motions caused by pouring to decay. Then a droplet of dye (alcohol solution of brilliant green) was introduced to the surface of the water with a pipette. In the water the droplet immediately formed a little cloud from which dye threads or wisps were pulled out. For control, a similar droplet was introduced into ordinary water where the usual molecular diffusion was observed but the dye remained mainly in the upper layer of the water.

Two typical times were measured: the time T_1, when a dye thread first touched the flask bottom, and the time T_m, when many threads, spaced more or less uniformly, were touching the bottom. The first time could be determined more or less distinctly, but the second time was determined rather subjectively, though with an accuracy up to 5 seconds which corresponds to about 15–20% from the value of T_m.

Two series of such measurements have been performed with mineral water from different bottles. The depth of the water varied from 2 to 9 cm. The results showed that both times, T_1 and T_m, with the accuracy indicated, did not, in fact, depend on the depth nor on the bottle. Knowing the depth and time, one can determine maximum and mean velocities, $u_1 = d/T_1$ and $u_m = d/T_m$. These velocities are shown in the lower part of figure 3, where different symbols relate to water from different bottles. The mean velocities of motion varied from 0·7 to about 4 mm s^{-1}, the maximum velocities being about twice as large.

For the shallowest layer, the Reynolds number calculated by u_m, d and

$$\nu = 0{\cdot}01\,\mathrm{cm^2\,s^{-1}}$$

is near 13, and for the deepest one, it is about 300. The lack, in our measurements, of any systematic dependence of time scale τ on the water depth can be regarded as a confirmation of the theory presented here and of the analogy between the two types of convection, thermal and density.

† Originally the pepper was tried as a tracer, but after some observations it was realized that the upward motion of the particles was caused mainly by gas bubbles attached to the particles or caught by them. The downward motion of the particles seemed to be from the elastic rebound of a rising particle from the water surface film. Accordingly, neither type of motion reflects the true fluid velocities.

7. Applications to ocean and mantle

The theory presented here has many applications to the ocean, but only a couple will be mentioned and no detailed calculations given The most evident region of application is the cooling of the surface layer of the ocean by a colder atmosphere. A cold film forms with thickness δ determined by (3.30) and with a temperature drop ΔT according to (3.31). The heat flux from the ocean is used in evaporation, thermal radiation and excitation of convection in the air. If the temperature difference between ocean and atmosphere is known then all the components of the energy budget in the absence of wind (or at weak winds below about $2\,\mathrm{m\,s^{-1}}$, Ginzburg & Fedorov 1978) can be estimated by the successive approximations described in § 4. Having the heat flux lost by the ocean, we can estimate the downward mass flux of cold water from equation (6.12): $M_t = \overline{\rho' w'} \approx \alpha f/c_p$. This value should be taken into account, for instance, while studying the rate of cooling of the mixed upper layer of the ocean. If as f we take $200\,\mathrm{W\,m^{-2}}$, a mean value of the solar radiation flux reaching the surface, then $\mathscr{M}_t \approx 10^{-6}\,\mathrm{g\,cm^{-2}\,s^{-1}} = 300\,\mathrm{kg\,m^{-2}\,yr^{-1}}$.

The theory of density convection (§ 5) can be used for studying the salt balance in the near surface layer of the ocean. A surplus of salt in this layer can arise because of evaporation and ice formation. Heating may also be important and estimates of the effects of the combined convection should use (6.13) and (6.14).

My interest in the convection problem arose originally from attempts to understand motions in the Earth's upper mantle which cause movements of the lithospheric plates. The aim was to obtain a simple formula for the velocities of the motion, to provide an elementary hydrodynamical base for further geophysical considerations. Such an estimate without the dependence on Nusselt number was first published in the author's paper (1977*a*). An intensive search through the convection and geophysical literature showed afterwards that such a formula but without estimates of the numerical coefficient was actually in the paper by Turcotte & Oxburgh (1967) [see their second formula (3.25)].

Because in our theory, the parameters of the medium are considered as constants, after McKenzie *et al.* (1974), it can be considered only as a very first approximation for a description of mantle convection. In reality, the viscosity of the medium depends very strongly on the temperature and pressure (Carter 1976), and this, in turn, influences strongly the flow patterns and their intensity. For instance, computations of thermal convection in the mantle by Houston & DeBremaecker (1975) with Herring–Nabarro viscosity (see Carter 1976) depending exponentially on temperature gave a noticeable rise of the convection velocity in the regions of lowered viscosity and a decrease where the viscosity is large. Nevertheless, these computations, as well as experiments by Booker (1976) carried out with a special oil with a viscosity strongly temperature dependent, show that the overall character and intensity of convective flows do not differ too drastically from the case of convection in fluid with constant parameters. One may hope, therefore, that our results will give the right orders of magnitude for the velocity and time scale of convection if one uses some effective value of viscosity. However the problem does require additional studies, numerical or laboratory.

The values of the thermal convection velocities in the mantle obtained here, as well as in the numerical experiments, which is of order $1\,\mathrm{cm\,yr^{-1}}$ seems to be insufficient, since many lithospheric plates are moving several times faster. If one takes into account the facts that the plates are moving as a whole, dragging each other, with oceanic plates diving under continental ones, one would feel safer if the mantle motions had velocities, say, of order $10\,\mathrm{cm\,yr^{-1}}$.

The structure of the formula (3.6) shows that this may be attained by an increase in the coefficient of thermal expansion α and/or by a decrease in the dynamic viscosity. Hewitt *et al.* (1975) note that the value of α is rather uncertain and could, in principle, be increased by an order of magnitude, which would increase the velocities by a factor of 3. However, this would also mean that the efficiency $\gamma \approx d/H = \alpha g d/c_p \sim 1$. But at $d \sim H$ the Boussinesq approximation breaks down and one should use, at least, the equations of deep convection. One should also not exclude the possibility that the value of viscosity $\nu = 2\times 10^{17}\,\mathrm{m^2\,s^{-1}}$, or $\mu = 7{\cdot}4\times 10^{21}P$, adopted by McKenzie *et al.* (1974) and here, is also considerably overestimated (see also Carter 1976). Accordingly, it *is* possible, even for purely thermal convection in the upper mantle, that $\bar{u}$ may be as large as $10\,\mathrm{cm\,yr^{-1}}$.

Concluding the discussion of this topic let us stress again that the self-similarity of convection at low Reynolds and high Prandtl numbers found in § 3, places laboratory modelling of mantle thermal convection, including modelling the flow patterns, on firm physical ground.

Let us turn now to a consideration of density gravitational convection in the mantle, though there are arguments against such a type of convection (e.g. McKenzie *et al.* 1974). Without going into detail of the geophysics of these questions, we present here only some estimates of the intensity of such a convection and point out some constraints.

As was shown in § 6, since the observed geothermal flux may only partly arise from purely heat sources, such as the heat of radioactive decay, the inequality (6.15) follows $\mathscr{M} < f/gd$. Consider the whole Earth's mantle with thickness $d = 3000\,\mathrm{km}$, in accordance with the belief that the phase differentiation takes place at the mantle–liquid core interface (Artyushkov 1968; Keondjan & Monin 1975, 1977). Then we obtain $\mathscr{M} < 2\times 10^{-9}\,\mathrm{kg\,m^{-2}\,s^{-1}} = 60\,\mathrm{gm\,m^{-2}\,yr^{-1}}$. During a period of time t_0, the density of the mantle material will be changed by

$$\Delta\rho \approx r_t t_0/d = \mathscr{M}t_0/d < f t_0/gd^2. \tag{7.1}$$

We neglect here the non-uniformity of the differentiation rate during the Earth's evolution – in the model of evolution by Monin & Keondjan (1976) this rate for the last four billion years changes by less than a factor of two. Taking the present value of f and $t_0 = 4$ Aeons, one gets from (7.1) that $\Delta\rho < 100\,\mathrm{kg\,m^{-3}} = 0{\cdot}1\,\mathrm{gm\,cm^{-3}}$. This value could be increased somewhat if one assumes that some part of the heat released in density convection goes to the heating of the mantle and to the support of the differentiation reactions which are, evidently, endothermic. Then, instead of (5.15), we should write

$$\mathscr{M}gd < f + Q, \tag{7.2}$$

where Q is the heat power spent in a unit column for heating the mantle and for the support of the reactions. Nevertheless, the value $\Delta\rho \approx 1\,\mathrm{g\,cm^{-3}}$ adopted by some investigators seems to be too high not only from the point of view of the constraints (6.15) or (7.2), but also in terms of estimates of the energy released by the gravitational density differentiation. In fact, according to Monin & Keondjan (1976) and several other models, the total energy released during the process is of order $1{\cdot}5 \times 10^{31}\,\mathrm{J}$ for the whole Earth's history. If all this energy were brought to the surface by convection uniformly, then the geothermal flux would be of order $0{\cdot}2\,\mathrm{W\,m^{-2}}$, three times the present value. Therefore an increase of $\Delta\rho$ of more than $0{\cdot}3\,\mathrm{g\,cm^{-3}}$ is difficult to attain. The excess energy could go only to the heating of the core. If the core mass is of the order $10^{25}\,\mathrm{kg}$ then the heating over 4 Aeons would be about 2000 K, although it would be less if part of the energy went for the support of the differentiation reactions.

For illustrative purposes we present estimates of mean velocities of density convection in the whole mantle. Let $\mathscr{M} \approx 2 \times 10^{9}\,\mathrm{kg\,m^{-2}\,s^{-1}}$. For a dynamic viscosity of the lower mantle $\mu \sim 10^{27}\,\mathrm{kg\,m^{-1}\,s^{-1}} = 10^{26}\,P$ (see McKenzie *et al.* 1974) one obtains from (3.6), replacing f by $c_p\mathscr{M}/\alpha$, that $\bar{u} \sim 1\,\mathrm{mm\,yr^{-1}}$. Keondjan & Monin (1977) assumed a value of μ smaller by three orders of magnitude as representative for the whole mantle. Then $\bar{u} \sim 3\,\mathrm{cm\,yr^{-1}}$. To have $\bar{u} \sim 10\,\mathrm{cm\,yr^{-1}}$ one should have justification to increase the ratio α/μ by yet another order of magnitude if one wishes to preserve the concept of convection in the whole mantle.

8. An attempt to classify forced geophysical flows

In § 3 a close analogy was noted between the convection theory presented for the viscous regime and turbulence structure in the dissipation subrange in the sense of the dependences on the external parameters forcing the flows [compare equations (3.2) and (3.4)]. It appears that thermal and density convection over a certain interval of Reynolds and Rayleigh numbers and turbulence in the dissipation range form a family of flows controlled by viscosity and by the power, thermal or mechanical, introduced into the fluid.

While working on convection, the author had in mind another type of forced flow, the circulation of planetary atmospheres. The first estimates of convective velocities (Golitsyn 1977*a*) were obtained along the line of the similarity theory for atmospheric circulations (Golitsyn 1970, 1973). In both the flow is controlled by viscosity and the atmospheric circulation on a moderately rotating planet (see also Bourangulov & Zilitinkevich 1976), there is one common feature, i.e. the total kinetic energy of both the atmospheric circulation and of the viscous fluid convection does not depend on the mass of the flow.

For convection, this follows from (3.1). In fact, if one considers a unit column of the viscous convecting fluid, then

$$K = \tfrac{1}{2}\rho d U^2 \approx \frac{\rho d}{2}\frac{d^2 f}{a^2 \rho \nu H}\frac{N-1}{N} = \frac{f d^3}{a^2 \nu H}\frac{N-1}{2N}. \tag{8.1}$$

Since the Nusselt number $N = fd/\rho c_p k\Delta T = fd/\lambda\Delta T$, where λ is the coefficient of heat conductivity, is independent of the properties of the medium, equation (8.1) demonstrates the independence of the kinetic energy K of the column from its mass.

For turbulence in the dissipation subrange, one should consider a volume with size less than Kolmogorov's microscale $d < \eta$, determined by (3.3), and calculate its kinetic energy K' relative to its immediate environment, expressing the total mechanical forcing in the volume as $\mathrm{E} = \rho\epsilon d^3$. The value of K' is then found also to be independent of the mass of the volume.

The kinetic energy of the global atmospheric circulation is of order

$$K \approx 2\pi\sigma^{\frac{1}{8}} c_p^{-\frac{1}{2}} q^{\frac{7}{8}} r^3, \tag{8.2}$$

where σ is the Stefan–Boltzmann constant, $q = \frac{1}{4}q_0(1-A)$ is the mean intensity of the solar radiation flux reaching the planet with albedo A, and r is the planetary radius (Golitsyn 1970, 1973). The independence of the kinetic energy of the flow from its mass is another feature which might be used for classification. This property is a reflexion of the insignificance of advection nonlinear terms in the momentum equation for an overall energy balance. For convection at $P \gg 1$, it is evident and it remains valid until the regularities of the viscous regime take over. For planetary circulations, this property follows from geostrophy of the motion and the dependence of the scale of large-scale motions on the Coriolis parameter (details can be followed in the paper by Bourangulov & Zilitinkevich 1976). However a number of forced flows do not possess this property, but together with the ones mentioned here, they have another more general feature which we shall now discuss.

Equation (8.2), after some simple transformations, can be rewritten as

$$K \sim Qr/c_e, \tag{8.3}$$

where $Q = 4\pi r^2 q$ is the total power of the solar energy assimilated by a planet, $c_e = (RT_e)^{\frac{1}{2}}$ is the isothermal sound velocity and $T_e = (q/\sigma)^{\frac{1}{4}}$ is the equilibrium radiation temperature of the planet. The quantity $\tau_e = r/c_e$ is the shortest time for propagation of a perturbation in the atmosphere over a global scale. It is known (see Landau & Lifshits 1954) that a state of local thermodynamical equilibrium is reached in a system of size d in a time d/c. Therefore to within a multiplier of order unity, the total kinetic energy of the circulation is

$$K \sim Q\tau_e, \tag{8.4}$$

that is, it is equal to the total radiative power assimilated by a planet times the shortest time of perturbation relaxation on a global scale.

But for the factors efficiency of convection γ times a^{-2}, both types of convection, thermal and density, have the same structure for the formulas for the total kinetic energy of a unit column. For thermal convection it has been already established by equation (4.4). For the density convection using the analogy established, one can also write

$$K \approx a^{-2}Q_m\tau_\nu,$$

where $Q_m = \gamma_g \mathscr{M} gd$ is the mechanical power introduced into the system and $\tau_\nu = d^2/\nu$ is the viscous relaxation time.

The same structure appears in the corresponding formula for the relative kinetic energy of a volume with size $d < \eta$. In this case $Q_m = \epsilon M = \rho\epsilon d^3 = \epsilon_1 d^3$ where ϵ_1 is the rate of dissipation of kinetic energy of turbulence per unit volume.

A corresponding form can also be given to the expression for the kinetic energy of a volume of locally isotropic and homogeneous turbulence relative to a neighbouring fluid volume of the same size $\eta < d < L_e$, where L_e is the turbulence external scale, so that d lies in the inertial interval (Kolmogorov 1941). The energy of such a volume with the mass $M = \rho d^3$ is

$$K \approx M(\epsilon d)^{\frac{2}{3}} \approx Q_m d/(\epsilon d)^{\frac{1}{3}}, \tag{8.5}$$

where $Q_m = M\epsilon$ is the total energy flux into the volume. In the inertial range of turbulence $(\epsilon d)^{\frac{1}{3}} \approx U$, the relative r.m.s. velocity for two points separated by the distance d, and

$$K \approx Q_m d/U = Q_m \tau_v, \tag{8.6}$$

where τ_v is a characteristic lifetime of eddies with scale d.

Note, however, that locally isotropic and homogeneous flow in the *inertial* subrange does not belong to the class of flows whose kinetic energy (in our relative sense) does not depend on the fluid mass: since $\epsilon = Q/M$, the energy, from (8.5), is proportional to $M^{-\frac{2}{3}}$. An analogous dependence of the total kinetic energy on mass has been also obtained in some models of general circulation, considered as large-scale convection on slowly rotating planets (Leovy & Pollack 1973; Burangulov & Zilitinkevich 1976) and in these models one may also obtain formulas of the type (8.5). The turbulent convection considered in § 5 also belongs to this type of flow. Together with the circulation models and turbulent flow in the inertial range it forms a family of flows with mean velocities proportional to $(\epsilon d)^{\frac{1}{3}}$. This class of flows may be called a family of forced flows controlled by inertia forces and turbulent mixing.

We see, therefore, that for quite a number of forced geophysical flows their total kinetic energy is determined by the product of the total power brought into the fluid and a characteristic relaxation time. Note that in the cases considered here this time is always the smallest of all the times which can be constructed from the external parameters of the problem. It is true that this time is often unique if one believes in the validity of the corresponding self-similarity hypotheses, which results in the exclusion of some dimensional parameters. Nevertheless, these examples allow one to propose the following approximate rule which could be called a 'principle of the fastest response': the kinetic energy of a forced steady flow of a fluid system is of the order of the total power brought into the system times the shortest relaxation time characteristic of the system.

If one is not using similarity theory, this rule allows one to write at once an expression for the total kinetic energy of the motion. It was this 'principle' that was noted in 1970 for the general circulation and was used in a first attempt to obtain an expression of the type (4.4), but without accounting for the convection efficiency γ and the multiplier a^{-2}.

We see that in the flows considered here the adherence to the 'principle of the fastest response' is their most general property. However, it is not universally true. An example of a system where it gives, at best, a lower bound of the kinetic energy is provided by the circulation in atmospheres of large and rapidly rotating planets such as Jupiter and Saturn. A detailed discussion of their circulation was given by Golitsyn (1970, 1973). It seems that the rapid rotation is a factor, strongly stabilizing large-scale motions and preventing the system from relaxing in the fastest way. Nevertheless, it appears that this 'principle' may have sometimes a heuristic value, as it has had in this study for which it served as a very first insight.

During the long work on this subject I discussed various aspects with many people and I can thank here only some of them. The first, who brought my attention to upper mantle convection in October 1970, was V. P. Troubitsyn who suggested that I seek to apply similarity arguments to this problem. Discussions with him, V. N. Zharkov and later, with P. N. Kropotkin helped my understanding of the problem. An impulse for returning to the problem after six years was derived from reading the paper by

Hide (1977) where there was a reference to McKenzie *et al.* (1974). Of much use and importance for me were several reprints which D. P. McKenzie sent to me and I appreciate very much his quick and kind reaction to my questions. Various parts of the work were presented in many seminars and the comments of many people were helpful and elucidating.

I am especially thankful to G. I Barenblatt for many discussions of theoretical problems and to A. S. Gurvich, Yu. L. Chernous'ko and K. N. Fedorov for advice and discussion of the experiments. Finally, I wish to thank my wife, Ludmila, for her help and patience with my home experiments and with my long and numerous attempts at writing and rewriting this paper.

Appendix. Similarity and dimensional arguments in application to convection in the viscous regime

A derivation of the formula for the mean kinetic energy of convective motions from similarity and dimensional arguments at small Reynolds numbers, but without the multiplier $1-N^{-1}$, is in the authors paper (1977*a*). Convection of a viscous fluid presents an instructive example of a concise use of modern ideas of similarity and dimensional theory, and this will be demonstrated in this section.

For simplicity we restrict ourselves by the case $Re \ll 1$ and neglect the internal heat sources and the viscous dissipation in the energy equation. Then the system (1.8), (1.9) is

$$\nu H \Delta \boldsymbol{\omega} = \nabla e \times \mathbf{n}, \quad \boldsymbol{\omega} = \nabla \times \mathbf{v}, \tag{A 1}$$

$$de/dt = k\Delta e \tag{A 2}$$

with boundary conditions of $\mathbf{v} = 0$ at the surface and bottom, say, and

$$k\partial e/\partial z = -f/\rho \quad \text{at} \quad z = 0 \quad \text{and} \quad e = 0 \quad \text{at} \quad z = d. \tag{A 3}$$

The system (A 1)–(A 3) has certain group properties. They were noticed by G. I. Barenblatt who proposed the idea of the following derivation of our formula (3.6). The structure of our system allows one to consider the dimension of the enthalpy as arbitrary, but then the dimensions of the other external parameters will be as follows: $[\nu H] = [b] = [e]LT$ [note that in (A 1) the combination $\nu H = b$ enters and because we neglect the inertia forces and viscous heating, the viscosity ν itself does not enter], $[f/\rho] = [e]LT^{-1}$, $[d] = L$, $[k] = L^2T^{-1}$. Here $[e]$ is the dimension of enthalpy, L and T are dimensions of length and time.

From first three quantities one can construct a quantity with dimension of velocity:

$$U = (fd^2/\rho\nu H)^{\frac{1}{2}} = (f/\mu H)^{\frac{1}{2}} d. \tag{A 4}$$

If one considers the fourth parameter, the thermal diffusivity k, then from all these quantities one may construct one non-dimensional combination, the Péclet number

$$Pe = Ud/k = (f/\mu H)^{\frac{1}{2}} d^2/k = R_f^{\frac{1}{2}}. \tag{A 5}$$

If $Pe = R_f^{\frac{1}{2}} \gg 1$ then the actual value of k is not important and we can neglect it in the set of determining parameters. This provides another derivation of equations (3.6) or (A 4). Evidently this derivation alone is rigorous only at $P = \nu/k \gg 1$. If the Rayleigh flux number R_f is not very large, then (A 4) should be multiplied by a function $f(R_f) = f(N(R_f))$. It is evident that $\lim_{x\to\infty} f(x) = 1$. Therefore we may expand $f(N)$ in a Taylor series near infinity and take only the first term $f(N) = 1-(a/N)$. It is also clear that as $N \to 1, f(N) \to 0$, because at $N = 1$ there is no motion. A function fulfilling the both requirements is $f(N) = 1-N^{-1}$. That this is really the correct function has been demonstrated in § 2. So with some physics in mind, it is even possible to reconstruct approximately the form of the function of the non-dimensional similarity criterion. Of course, such a reconstruction came to mind after the exact results had been obtained by other means, but some idea of its behaviour could, in principle, be developed without it.

The possibility of an arbitrary choice for the enthalpy dimension may be demonstrated by the following arguments. The system (A 1)–(A 3) is invariant relative to transformations

$$e \to \alpha e', \quad \nu H \to \alpha(\nu H)', \quad f/\rho \to \alpha(f/\rho)', \tag{A 6}$$

where α is an arbitrary number. If one assumes, as usual, that $[e] = L^2T^{-2}$, then $[f/\rho] = L^3T^{-3}$ and $[\nu H] = L^3T^{-1}$. Then the three first external parameters can form a non-dimensional quantity $\Pi = (f/\rho)\, d^2(\nu H)^{-\frac{3}{2}}$. Under the transformation (A 6), this quantity changes as $\Pi \to \alpha^{\frac{2}{3}}\Pi'$, i.e. it depends on the choice of the value α, in contradiction to our main system (A 1)–(A 3). This argument shows that the dimension of enthalpy should be considered as arbitrary.

Owing to the importance of the convective time scale τ we shall derive it also from similarity and dimension arguments. In a general case, including the case when $Re \gtrsim 1$, the equation system (1.8)–(1.10) has five dimensional external parameters: $[H] = L$, $[\nu] = L^2T^{-1} = [k]$, $[f/\rho] = L^3T^{-3}$ and $[d] = L$. Only the dimensions of time and length enter into them, and therefore one can construct three non-dimensional combinations and we choose the Rayleigh number $R_f = fd^4/\rho\nu k^2 H$, the Prandtl number $P = \nu/k$ and $\gamma_0 = d/H$. Then the time scale will be a function of these three similarity parameters and the form of the function will depend on the way that we construct the time scale from these five external parameters. There are many possibilities. Let us choose

$$\tau = (f/\rho\nu H)^{\frac{1}{2}}\, \psi(R_f, P, \gamma_0). \tag{A 7}$$

Various experiments and estimates show that there is an interval of rather large values of R_f, not very small P and small γ_0 when the function $\psi(R_f, P, \gamma_0)$ tends to a constant, approximately equal to 12–13. If we know the dependence of velocity on R_f, more precisely, on N, then one can obtain the dependence on N in the viscous regime, arguing that for $N \to 1$ the value of τ should tend to infinity and for large N, to a constant.

For the standard choice of the time scale as d^2/k we should have

$$\tau = (d^2/k)\, \psi_1(R_f, P, \gamma_0)$$

and to obtain the correct result we should assume self-similarity on P and γ_0 but assume that $\psi_1(R_f) \sim R_f^{-\frac{1}{2}}$ for sufficiently large R_f (outside the turbulent regime). We see that the time scale d^2/k is less natural than in (A 7) because it is clear that for large R_f, the action of molecular thermal conductivity is restricted only to thin thermal boundary layers; in the bulk of fluid it should be insignificant and should not enter a set of determining parameters.

In § 2 it is shown for sufficiently large Rayleigh numbers, when the Nusselt number is large, the convection efficiency γ is d/H. From the point of view of similarity theory this result may be understood as follows. Accounting for dissipation means the appearance of one more dimensional parameter in the energy equation (A 2), i.e. the kinematic viscosity ν by itself. Adding it to the existing parameters $b = \nu H$, d and f/ρ gives the possibility of forming a new non-dimensional similarity parameter $\gamma = \nu d/b = d/H$. In relation to dissipation, we have a typical case of self-similarity of the second kind, in the terminology of Barenblatt (1978). The r.m.s. convective velocities (A 4) are self-similar relative to the similarity parameters *Re*, *Pe* and γ; the self-similarity with respect to *Re* and γ mean their independence from ν. If we accept a similar hypothesis for the dissipation rate ϵ, then from the dimensional parameters $\nu H, f/\rho$ and d, one could construct a quantity with the dimension of ϵ only as $a_1 f/\rho d$, where $a_1 =$ const. This was done in the first version of the author's (1977*a*) paper, where the value of a_1 was determined from computations by McKenzie *et al.* (1974). Neglecting the heating from viscous dissipation means the neglect of the similarity parameter $\gamma = \nu d/b$. However, this can be done while determining the dissipation from similarity arguments if only $\lim_{\gamma\to 0} a_1(\gamma) =$ const. But here

$$\lim_{\gamma\to 0} a_1(\gamma) = \nu d/b = d/H = \gamma_0, \tag{A 8}$$

which is a characteristic for self-similarity of the second kind (Barenblatt 1976, 1978), when a certain quantity, depending on a non-dimensional parameter in its approach to zero or infinity, does not tend to a finite limit but changes with it by some power law. The character of the latter dependence cannot be determined from similarity and dimensional arguments only, but some other considerations should be used; here, the formula (A 4) for the dissipation rate. The dependence of the type $1 - N^{-1}$ as a multiplier in (A 8) may be obtained again by matching asymptotics at $N \to \infty$ and $N \to 0$.

Let us mention another evident circumstance: if the horizontal and vertical scales of the convection region are different then one more similarity criterion, the aspect ratio, appears. If there are internal heat sources, their distribution and intensity should be characterized by a parameter of the type β, see equation (2.12).

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Note added in proof (25 September 1979)

S. Chandrasekhar (*Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, 1961) has calculated the velocity fields for the roll convection in the limit of weak nonlinearity. Combining his results with ours on the efficiency of convection one can prove that our formula for the kinetic energy is exact in this limit. Moreover this energy can be calculated from his results producing $a = 8{\cdot}81$.